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(G,D) - Number of Weak Product and Composition of Graphs

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Abstract:- **(G,D)-number of graphs was introduced by Palani K and Nagarajan A. Let G be a (V,E) graph. A dominating set is a subset D of V such that every vertex in V-D is adjacent to atleast one vertex of D. A (G,D)-set D of G is a subset D of V(G) which is both a dominating and a geodetic set of G.In this paper, we find the (G,D)-number of weak(or kronecker) product and strong product (or composition)of some standard graphs.**

I. INTRODUCTION

Graph Theory is an important branch of Mathematics. It has grown rapidly in recent times with a lot of research activities. In 1958, domination was formulized as a theoretical area in graph theory by C. Berge. He referred to the domination number as the coefficient of external stability and denoted as β(G)**.** In 1962, Ore [7] was the first to use the term 'Domination' number by δ(G) and also he introduced the concept of minimal and minimum dominating set of vertices in graph. In 1977, Hedetniemiet.al[6] introduced the accepted notation γ(G) to denote the domination number. Let G = (V,E) be any graph. A dominating set of a graph G is a set D of vertices of G such that every vertex in V-D is adjacent to atleast one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number of G. It is denoted by γ(G).The concept of geodominating (or geodetic) set was introduced by Buckley and Harary in [2] and Chartrand, Zhang and Harary in [3, 4,5]. Let u, v ∈V(G). A u-v geodesic is a u-v path of length d(u, v). A vertex x is said to lie on a u-v geodesic p if x is any vertex on p. A set S of vertices of G is a geodominating (or geodetic) set if every vertex of G lies on an x-y geodesic for some x,y in S. The minimum cardinality of geodominating set is the geodomination (or geodetic) number of G. It is denoted by g(G). K. Palani et.al[8,9,10] introduced the concept (G,D)-set of graphs. A (G,D)- set of graph G is a subset S of vertices of G which is both dominating and geodominating (or geodetic) set of G. A (G,D)- set of G is said to bea minimal (G,D) set of G if no proper subset of S is a (G,D)-set of G. The minimum cardinality of all minimal (G,D)-set of G is called the (G,D)- number of G. It is denoted by $\gamma_G(G)$. All graphs considered here are non-trivial, simple and undirected. The order and size of a graph G are denoted by p and q respectively Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be two graphs. The***weak(or kronecker) product*****[1]** of G_1 and G_2 denoted by$G_1 \odot G_2$has $V = V_1 \times V_2$ as its vertex setand E $=\{\{(u_1,u_2)(v_1,v_2)\}/(u_1,v_1) \in E_1$ and $(u_2,v_2) \in E_2\}$as its edge set.The ***strong(or composition) product***[1] of G_1 and G_2 is denoted by $G_1 \boxtimes G_2$ has $V = V_1 \times V_2$ as its vertex set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever u_1 is adjacent to v_1 (or) $u_1 = v_1$ and u_2 is adjacent to v_2. In this paper, we find the (G,D)-number of weak(or kronecker) product and strong product (or composition)of some standard graphs. The following theorems are from [8,9,10]

1.1 Theorem : $\gamma_G(K_n) = n$

1.2Theorem : $\boldsymbol{\gamma_G}(P_n) = 2 + \left\lceil \frac{n-4}{3} \right\rceil$

1.3 Theorem : $\gamma_G(C_n) = \left\lceil \frac{n}{3} \right\rceil$, $n \geq 6$.

1.4 Theorem: Any (G,D) set D of a Graph G contains all extreme points of G.In particular, D contains all the end points of G.

II. (G,D) - NUMBER OF WEAK(OR KRONECKER) PRODUCT OF GRAPHS

Here, we find the (G,D)-number of weak product of some standard graphs.

2.1 Theorem : $\gamma_G(P_2 \odot P_n) = 2\left\lceil \frac{n+2}{3} \right\rceil$

Proof: Let $V(P_2) = \{u_1,u_2\}$ and $V(P_n) = \{v_1,v_2,...v_n\}$.
Then, $P_2 \odot P_n$ looks as in figure 2.1
Here, $P_2 \odot P_n \cong 2P_n$.
Therefore, $\gamma_G(P_2 \odot P_n) = \gamma_G(2P_n)$
$= 2\gamma_G(P_n)$
$= 2\left\lceil \frac{n+2}{3} \right\rceil$.

International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 4, April 2018

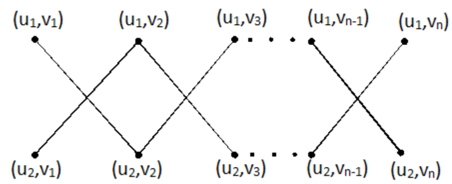

Figure 2.1

2.2 Theorem : $\gamma_G(P_3 \odot P_n) = n+2$ for $n \geq 4, n \neq 5$.

Proof: Let $n \geq 4$, $n \neq 5$. Let $V(P_3) = \{u_1,u_2,u_3\}$ and $V(P_n) = \{v_1,v_2,...v_n\}$.

Then, $P_3 \odot P_n$ looks as in figure 2.2

Obviously, $S = \{(u_1,v_1), (u_2,v_1), (u_3,v_1), (u_1,v_n), (u_2,v_n), (u_3,v_n)\}$ is a minimum geodetic set of $P_3 \odot P_n$ They dominate only the 6 vertices with second coordinate v_2 and v_{n-1}.

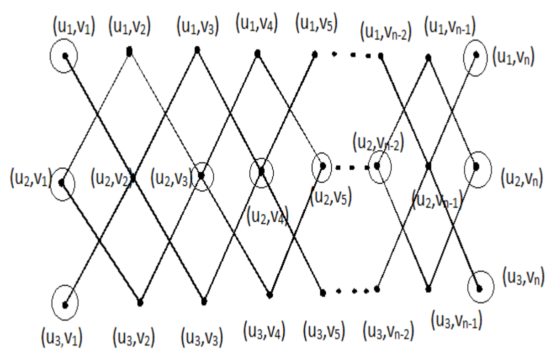

Figure 2.2

To dominate the remaining vertices we need to select either all the vertices in the second row with second co-ordinate not equal to v_1,v_2,v_{n-1} and v_n or atleast the same number of vertices from all the three rows to get a minimum (G,D)-set.

Hence, $S_1 = S \cup \{(u_2,v_3), (u_2,v_4),... (u_2,v_{n-2})\}$ is a minimum (G,D)-set of $P_3 \odot P_n$ Therefore, $\gamma_G(P_3 \odot P_n) = |S_1| = 6+n-4 = n+2$ for $n \geq 4$, $n \neq 5$.

When n = 5, the graph is as in figure. 2.3

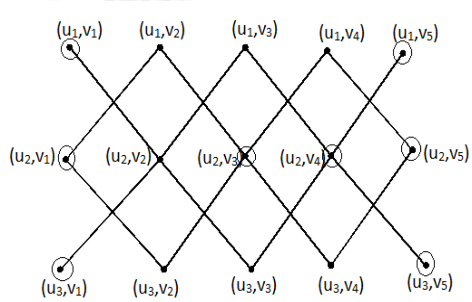

Figure 2.3

In this case $S_1 = \{(u_1,v_1), (u_2,v_1), (u_3,v_1), (u_1,v_5), (u_2,v_5), (u_3,v_5), (u_2,v_3),(u_2,v_4)\}$ (or) $S_2 = \{(u_1,v_1), (u_2,v_1), (u_3,v_1), (u_1,v_5), (u_2,v_5), (u_3,v_5), (u_2,v_3), (u_2,v_2)\}$ are the minimum (G,D)-sets.

Therefore, $\gamma_G(P_3 \odot P_n) = |S_1| = |S_2| = 8$.

2.3 Theorem :

$\gamma_G(P_4 \odot P_n)$

$$= \begin{cases} 4k+6 \ \ if \ n = 4k+1 \\ 4(k+1) \ if \ \ n = 4k+2, 4k+3, 4(k+1) \end{cases}$$

Proof: Let $V(P_4 \odot P_n) = \{u_1,u_2,...u_n,v_1,v_2,...v_n.w_1.w_2,...w_n,z_1,z_2,...z_n\}$ with u_i,v_i,w_i,z_i representing the corresponding row elements. Then, $P_4 \odot P_n$ is as in figure 2.4

Obviously, $S = \{u_1,v_1,w_1,z_1,u_n,v_n,w_n,z_n\}$ is a geodetic set of $P_4 \odot P_n$.

Also, they dominate the 8 vertices in the second and $(n-1)^{th}$ column elements.

Further, it is observed that selecting last two consecutive vertices among every four consecutive vertices from the second and third row starting with second column element, we could get a dominating set for the remaining elements.

Therefore, to find the vertices to dominate the remaining vertices of $P_4 \odot P_n$. We proceed in 4 cases

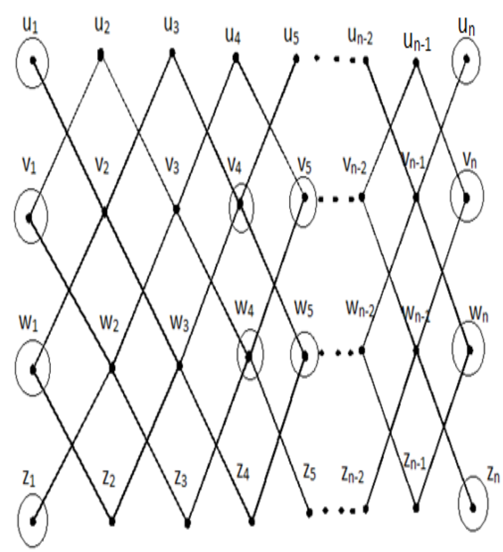

Figure 2.4

Case (i) If n = 4k + 1

Here, $S \cup \{v_4,v_5,w_4,w_5,v_8,v_9,w_9,...v_{4(k-1)},v_{4(k-1)+1},w_{4(k-1)},w_{4(k-1)+1},v_{4k},w_{4k}\}$ is a minimum (G,D)-set of $P_4 \odot P_n$ when n = 4k+1.

Therefore, $\gamma_G(P_4 \odot P_n) = |S| = 8 + 4(k-1) + 2$

$= 10 + 4(k-1)$

$= 4k + 6$.

Case (ii) If n = 4k + 2.

Here, $S \cup \{v_4,v_5,w_4,w_5,v_8,v_9,w_8,w_9,...v_{4k},v_{4k+1},w_{4k},w_{4k+1}\}$ is a minimum (G,D)-set of $P_4 \odot P_n$ when n = 4k+2.

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Vol 3, Issue 4, April 2018

Therefore, $\gamma_G(P_4 \odot P_n)$=|S| =8 + 4(k)=4(k + 2).

Case (iii) If n = 4k + 3.

Here, $S \cup \{v_4,v_5,w_4,w_5,v_8,v_9,w_8,w_9,...v_{4k},v_{4k+1},w_{4k},w_{4k+1}\}$ is a minimum (G,D)-set of $P_4 \odot P_n$ when n = 4k+3.

Therefore, $\gamma_G(P_4 \odot P_n)$=|S|= 8 + 4(k)= 4(k + 2).

Case (iv) If n = 4(k + 1).

Here, $S \cup \{v_4,v_5,w_4,w_5,v_8,v_9,w_8,w_9,...v_{4k},v_{4k+1},w_{4k},w_{4k+1}\}$ is a minimum (G,D)-set of $P_4 \odot P_n$ when n = 4(k+1).

Therefore, $\gamma_G(P_4 \odot P_n)$=|S|=8 + 4(k)= 4(k + 2).

2.4 Illustration

Consider $P_4 \odot P_5$ as in figure 2.5

Here, n = 5 = 4k + 1 where k = 1

S = { $u_1,v_1,w_1,z_1,u_5,v_5,w_5,z_5\} \cup \{v_4,w_4\}$ is a minimum (G,D)-set of $P_4 \odot P_5$.

Therefore, $\gamma_G(P_4 \odot P_5)$ = |S| = 10= 4k + 6.

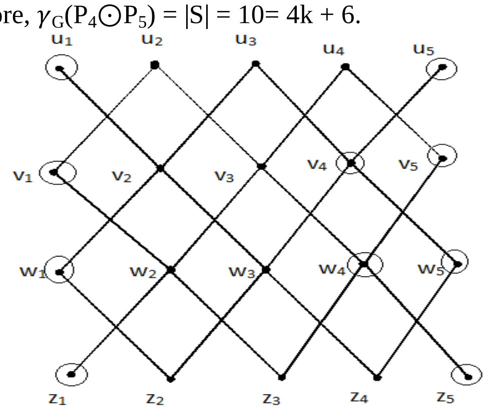

Figure 2.5

2.5 Illustration

Consider $P_4 \odot P_6$ as in figure 2.6

Here, n = 6 = 4k + 2 where k = 1

S = { $u_1,v_1,w_1,z_1,u_6,v_6,w_6,z_6\} \cup \{v_4,v_5,w_4,w_5\}$ is a minimum (G,D)-set of $P_4 \odot P_6$.

Therefore, $\gamma_G(P_4 \odot P_6)$ = |S| = 12= 4(k + 2).

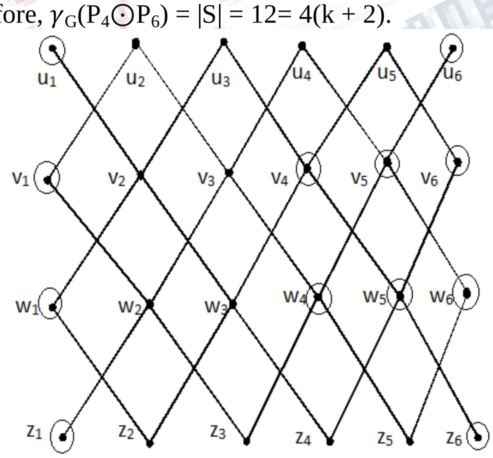

Figure 2.6

2.6 Illustraion

Consider $P_4 \odot P_7$ as in figure 2.7

Here, n = 7 = 4k + 3 where k = 1

S = { $u_1,v_1,w_1,z_1,u_7,v_7,w_7,z_7\} \cup \{v_4,v_5,w_4,w_5\}$ is a minimum (G,D)-set of $P_4 \odot P_7$.

Therefore, $\gamma_G(P_4 \odot P_7)$ = |S| = 12= 4(k + 2).

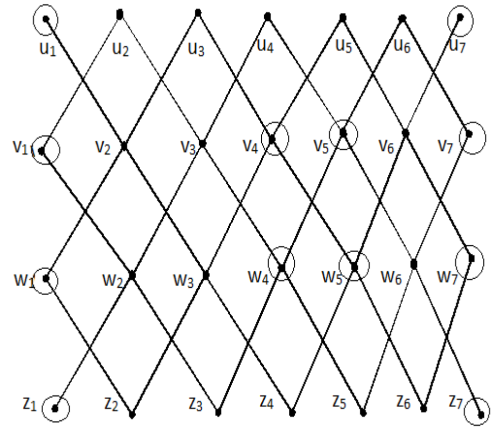

Figure 2.7

2.7 Illustration :

Consider $P_4 \odot P_8$ as in figure 2.8

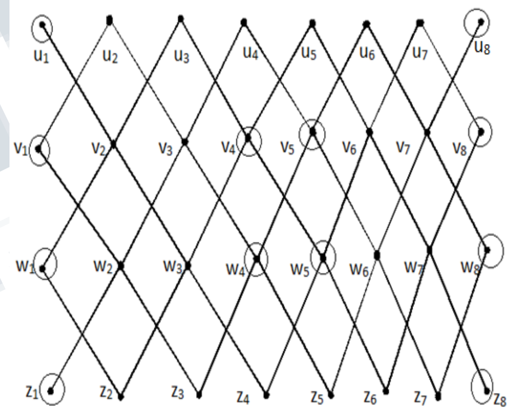

Figure 2.8

Here, n = 8 = 4(k + 1) where k = 1

S = { $u_1,v_1,w_1,z_1,u_8,v_8,w_8,z_8\} \cup \{v_4,v_5,w_4,w_5\}$ is a minimum (G,D)-set of $P_4 \odot P_8$.

Therefore, $\gamma_G(P_4 \odot P_8)$ = |S| = 12=4(k+2).

2.8 Theorem :

$$\gamma_G(C_2 \odot C_n) = \begin{cases} 2\lceil n/3 \rceil \, if \; n \; is \; even \\ \lceil 2n/3 \rceil \; if \; n \; is \; odd \end{cases}; \; n \geq 6.$$

Proof: Let $V(C_2) = \{u_1,u_2\}$ and $V(C_n) = \{v_1,v_2,...v_n\}$.

Then, $C_2 \odot C_n$ looks as in figure 2.9

$$\text{Here, } C_2 \odot C_n \cong \begin{cases} 2C_n \; if \; n \; is \; even \\ C_{2n} \; if \; n \; is \; odd \end{cases}.$$

Therecfore, $\gamma_G(C_2 \odot C_n)$

$$= \begin{cases} \gamma_G(2C_n) \; if \; n \; is \; even \\ \gamma_G(C_{2n}) \; if \; n \; is \; odd \end{cases}$$

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International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 4, April 2018

$$= \begin{cases} 2\gamma_G(C_n) \; if \; n \; is \; even \\ \gamma_G(C_{2n}) \; if \; n \; is \; odd \end{cases}$$

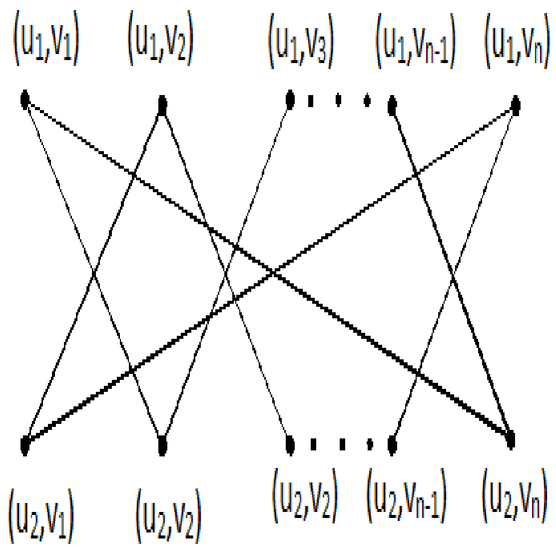

Figure 2.9

Therefore, by Theorem 1.3,

$$\gamma_G(C_2 \odot C_n) = \begin{cases} 2\lceil n/3 \rceil \; if \; n \; is \; even \\ \lceil 2n/3 \rceil \; if \; n \; is \; odd \end{cases}$$

2.9 Remark :

(1)$\gamma_G(C_2 \odot C_3) = \gamma_G(C_6) = \lceil 6/3 \rceil = 2.$

(2)$\gamma_G(C_2 \odot C_4) = 2\,\gamma_G(C_4) = 2 \times 2 = 4$

$= 2 \times \left\lceil \frac{4}{3} \right\rceil$

(3) By (1) and (2), it is observed that the above theorem is true for all values of $n \geq 3$ though $\gamma_G(C_n) = \lceil n/3 \rceil$ for $n \geq 6$.

2.10 Theorem :

$\gamma_G(C_3 \odot C_n) = n; \; n \geq 3$

Proof: Let $n \geq 3$. Let $V(C_3) = \{u_1,u_2,u_3\}$ and $V(C_n) = \{v_1,v_2,...v_n\}$.

Then, $C_3 \odot C_n$ looks as in figure 2.10Obviously, $S=\{(u_1,v_1), (u_1,v_2), (u_1,v_3), (u_1,v_4),... (u_1,v_{n-2}), (u_1,v_{n-1}), (u_1,v_n)\}$ or$\{(u_2,v_1), (u_2,v_2), (u_2,v_3), (u_2,v_4),... (u_2,v_{n-2}), (u_2,v_{n-1}), (u_2,v_n)\}$ or $\{(u_3,v_1), (u_3,v_2), (u_3,v_3), (u_3,v_4),... (u_3,v_{n-2}), (u_3,v_{n-1}), (u_3,v_n)\}$ is a minimum geodetic set of $C_3 \odot C_n$.

Also, they dominate all the remaining vertices in $C_3 \odot C_n$. Thus S is a minimum (G,D)-set of $C_3 \odot C_n$.Therefore, $\gamma_G(C_3 \odot C_n) = n; \; n \geq 3$

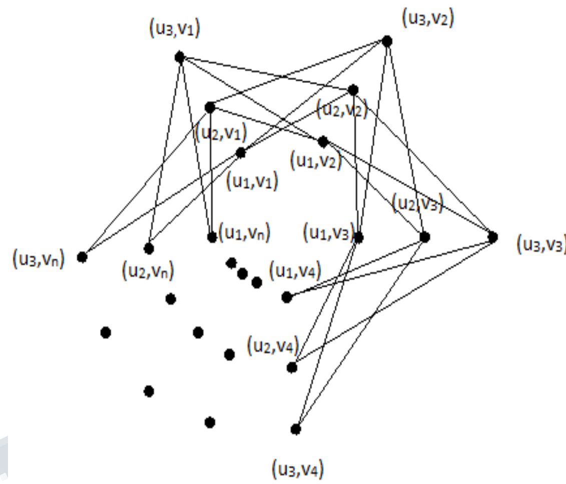

Figure 2.10

III. (G,D) – NUMBER OF STRONG(OR COMPOSITION) PRODUCT OF GRAPHS

Here, we find the (G,D)-number of strong product of some standard graphs.

3.1 Proposition :$\gamma_G(P_2 \boxtimes P_3) = 2.$

Proof: Let $V(P_2) = \{u_1,u_2\}$ and $V(P_3) = \{v_1,v_2,v_3\}$Then, $P_2 \boxtimes P_3$ looks as in figure 3.1

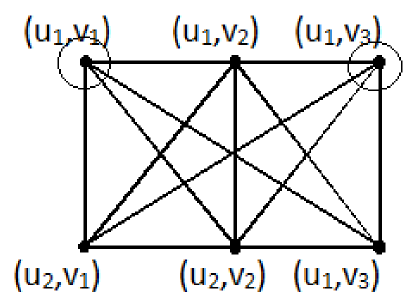

Figure 3.1

From figure, clearly $S = \{(u_1,v_1),(u_1,v_3)\}$ is one of the minimum (G,D)-set of $P_2 \boxtimes P_3$.

Therefore, $\gamma_G(P_2 \boxtimes P_3) = |S| = 2.$

3.2 Proposition : $\gamma_G(P_2 \boxtimes P_4) = 3.$

Proof:Let $V(P_2) = \{u_1,u_2\}$ and $V(P_4) = \{v_1,v_2,v_3,v_4\}$

Then,$P_2 \boxtimes P_4$looks as in figure 3.2

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Vol 3, Issue 4, April 2018

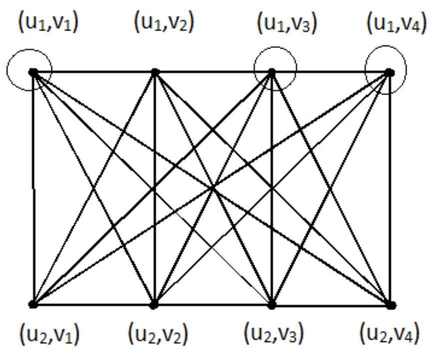

Figure 3.2

From figure,clearly $S = \{(u_1,v_1),(u_1,v_3),(u_1,v_4)\}$ is one of the minimum (G,D)-set of $P_2 \boxtimes P_4$.
Therefore, $\gamma_G(P_2 \boxtimes P_4) = |S| = 3$.

***3.3 Theorem* :** $\gamma_G(P_2 \boxtimes P_n) = 4$ where $n > 4$.
Proof: Label the vertices of $P_2 \boxtimes P_n$ as $u_1,u_2,...u_n$ and $v_1,v_2,...v_n$ as in figure 3.3
It is observed that $P_2 \boxtimes P_n$ contains a bipartite graph $K_{n,n}$ as its subgraph. Further, $V(K_{n,n})$ is partitioned into V_1, V_2 with $G[V_1]$, $G[V_2]$ are path of length n.
Hence V_1, V_2 contains atleast two non-adjacent vertices.Therefore, a pair of two non-adjacent vertices from V_1 with a pair of two non-adjacent vertices from V_2 forms a geodetic set.

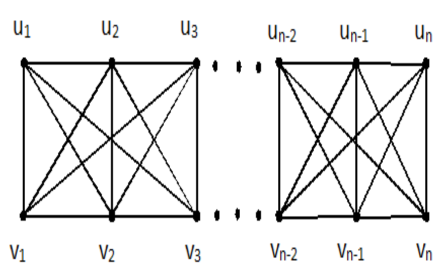

Figure 3.3

Further, $P_2 \boxtimes P_n$ contains $K_{n,n}$ implies the above is also a dominating set of $P_2 \boxtimes P_n$.
Hence $S = \{u_i,u_j,v_i,v_j / u_i,u_j$ are non-adjacent and v_i,v_j are non-adjacent$\}$ is a (G,D)-set of $P_2 \boxtimes P_n$.Further, S is a minimum if $n > 4$.
Hence $\gamma_G(P_2 \boxtimes P_n) = |S| = 4$.

3.4 Theorem : $\gamma_G(P_3 \boxtimes P_n) = 4$ where $n > 3$.
Proof: Label the vertices of $P_3 \boxtimes P_n$ as $u_1,u_2,...u_n$, $v_1,v_2,...v_n$,$w_1,w_2,...w_n$ as in figure 3.4

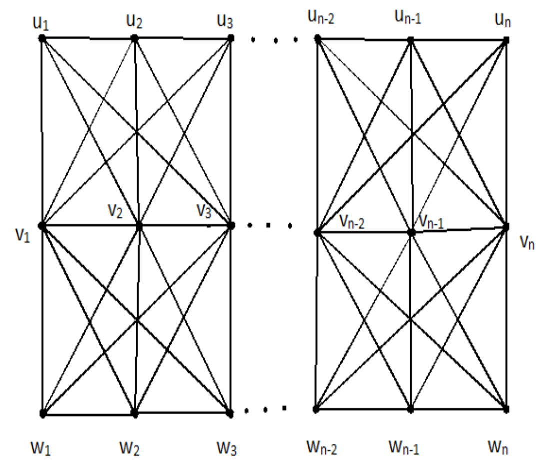

Figure 3.4

It is observed that $S = \{u_i,u_j,v_i,v_j / u_i,u_j$ are non-adjacent and v_i,v_j are non-adjacent$\}$ is one of the (G,D)-set of $P_3 \boxtimes P_n$.
Further, S is a minimum if m>3.Hence $\gamma_G(P_3 \boxtimes P_n) = |S| = 4$.
3.5Theorem : $\gamma_G(P_4 \boxtimes P_n) = 4$ where $n > 2$.
Proof: Label the vertices of $P_4 \boxtimes P_n$ as $u_1,u_2,...u_n$, $v_1,v_2,...v_n$,$w_1,w_2,...w_n$, $z_1,z_2,...z_n$ as in figure 3.5.
It is observed that $S = \{v_i,v_j,w_i,w_j / v_i,v_j$ are non-adjacent and w_i,w_j are non-adjacent$\}$is a (G,D)-set of $P_4 \boxtimes P_n$.
Further, S is a minimum if $n > 2$. Hence $\gamma_G(P_4 \boxtimes P_n) = |S| = 4$.

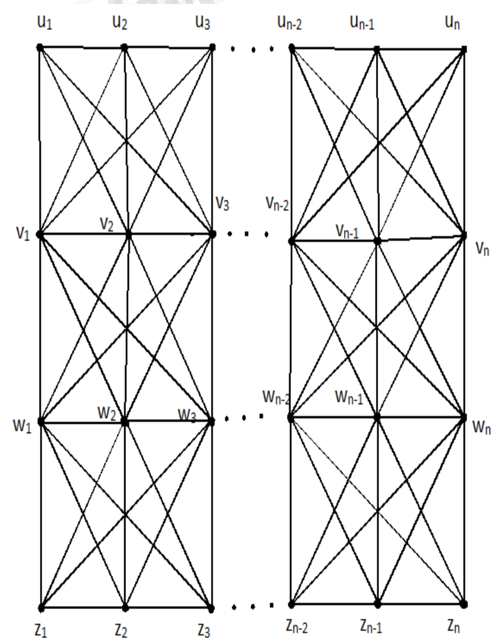

Figure 3.5

3.6 Theorem : $\gamma_G(C_2 \boxtimes C_2) = 4$.
Proof: Let $V(C_2) = \{u_1,u_2\}$ and $V(C_2) = \{v_1,v_2\}$
From figure 3.6, $C_2 \boxtimes C_2 \cong K_4$.

By Theorem 1.1, $\gamma_G(C_2 \boxtimes C_2) = \gamma_G(K_4) = 4$.

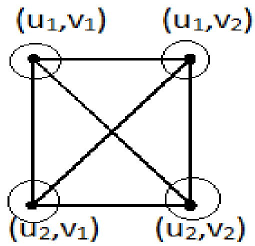

Figure 3.6

3.7 Theorem: $\gamma_G(C_2 \boxtimes C_3) = 6$.
Proof: Let $V(C_2) = \{u_1,u_2\}$ and $V(C_3) = \{v_1,v_2,v_3\}$
From figure 3.7, $C_2 \boxtimes C_3 \cong K_6$
By Theorem 1.1, $\gamma_G(C_2 \boxtimes C_2) = \gamma_G(K_6) = 6$.

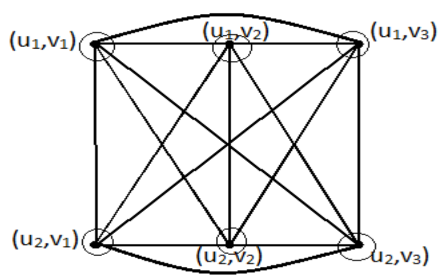

Figure 3.7

CONCLUSION:

In general, $C_2 \boxtimes C_n$ is need not be a complete graph. (G,D)-number of $C_m \boxtimes C_n$ for any two integers m and n could be investigated in a similar manner.

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