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On Semi*δ-regular and Semi*δ-normal Spaces

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Abstract:- **The purpose of this paper is to introduce the concepts of semi*δ-regular and semi*δ-normal spaces using semi*δ-open sets and investigate their basic properties. We also discuss their relationships with already existing concepts.**

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Keywords: semi*δ-regular, semi*δ-normal.

1. INTRODUCTION

Maheswari and Prasad [4, 5] first defined the notion of S-normal spaces by replacing open sets in the definition of normal spaces by semi-open sets. Dorsett [1, 2] introduced the concept of semi-regular and semi-normal spaces and investigate their properties. The purpose of this paper is to introduce the concepts of semi*δ-regular space, semi*δ-Normal Space and study their basic properties.

II. PRELIMINARIES

Throughout this paper(X,τ),(Y,σ) and (Z,η) will always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of (X,τ), Cl(A) and Int(A) denote the closure, the interior of A. We recall some known definitions needed in this paper.

Definition 2.1: A subset A of a topological space (X, τ) is called semi-open [3] (respectively semi*-open [11]) if A⊆Cl(Int(A)) (respectively A⊆Cl*(Int(A)) .
Definition 2.2: A subset A of a topological space (X, τ) is called semi*δ-open [6] (respectively semi*δ-closed [7]) if A⊆Cl*(δInt(A))(respectively Int*(δCl(A))⊆A).
Definition 2.3: A function f : (X, τ)⟶ (Y, σ) is said to be
(i) closed [12] if f (V) is closed in Y for every closed set V in X.
(ii) semi*δ-continuous[8] if f^{-1}(V)is semi*δ-open in X for every open set V in Y.
(iii) semi*δ-irresolute[8] if f^{-1}(V) is semi*δ-open in X for every semi*δ-open set V in Y.
(iv) semi*δ-open[9] if f (U) is semi*δ-open in Y for every open set U in X.
(v) pre-semi*δ-open[9] if f(U) is semi*δ-open in Y for every semi*δ-open set U in X.
(vi) pre-semi*δ-closed[9] if f(F) is semi*δ-closed in Y for every semi*δ-closed set F in X.
Definition 2.4: A space X is said to be T_1[12] if for every pair of distinct points x and y in X, there is an open set U containing x but not y and an open set V containing y but not x.
Definition 2.5: A topological space X is said to be
(i) regular if for every pair consisting of a point x and a closed set B not containing x, there are disjoint open sets U and V in X containing x and B respectively.[12]
(ii) s-regular if for every pair consisting of a point x and a closed set B not containing x, there are disjoint semi-open sets U and V in X containing x and B respectively.[4]
(iii) semi-regular if for every pair consisting of a point x and a semi-closed set B not containing x, there are disjoint semi-open sets U and V in X containing x and B respectively.[1]
(iv) semi*-regular if for every pair consisting of a point x and a semi*-closed set B not containing x, there are disjoint semi*-open sets U and V in X containing x and B respectively.[10]
(v) s*-regular if for every pair consisting of a point x and a closed set B not containing x, there are disjoint semi*-open sets U and V in X containing x and B respectively.[10]
Definition 2.6: A topological space X is said to be
(i) normal if for every pair of disjoint closed sets A and B in X, there are disjoint open sets U and V in X containing A and B respectively.[12]
(ii) s-normal if for every pair of disjoint closed sets A and B in X, there are disjoint semi-open sets U and V in X containing A and B respectively.[5]
(iii) semi-normal if for every pair of disjoint semi-closed sets A and B in X, there are disjoint semi-open sets U and V in X containing A and B respectively.[2]
(iv) semi*-normal if for every pair of disjoint semi*-closed sets A and B in X, there are disjoint semi*-open sets U and V in X containing A and B respectively.[10]
(v) s**-normal if for every pair of disjoint closed sets A and B in X, there are disjoint semi*-open sets U and V in X containing A and B respectively.[10]
Theorem 2.7: A function f: X⟶Y is semi*δ-irresolute if f^{-1}(F) is semi*δ-closed in X for every semi*δ-closed set F in Y. [8]

III. REGULAR SPACES ASSOCIATED WITH SEMI*δ-OPEN SETS.

Definition 3.1: A space X is said to be **semi*δ-regular** if for every pair consisting of a point x and a semi*δ-closed set F not containing x, there are disjoint semi*δ-open sets U and V in X containing x and F respectively.

Theorem 3.2: In a topological space X, the following are equivalent:

(i) X is semi*δ-regular.

(ii) For every x∈X and every semi*δ-open set U containing x, there exists a
semi*δ-open set V containing x such that s*δCl(V) ⊆U.

(iii) For every set A and a semi*δ-open set B such that A∩B ≠ ϕ, there exists a semi*δ- open set U such that A∩U≠ϕ and s*δCl(U) ⊆B.

(iv) For every non-empty set A and semi*δ-closed set F such that A∩F ≠ϕ, there exist disjoint semi*δ-open sets U and V such that A∩U≠ϕ and F⊆V.

Proof: (i) ⟹ (ii): Let U be a semi*δ-open set containing x. Then B=X\U is a semi*δ-closed set not containing x. Since X is semi*δ-regular, there exist disjoint semi*δ-open sets V and U containing x and B respectively. Then s*δCl(V) is disjoint from B, since if y ∈ B, the set U is a semi*δ-open set containing y disjoint from V. Hence s*δCl(V)⊆U

(ii) ⟹ (iii): Let A∩B≠ϕ and B be semi*δ-open. Let x∈A∩B. Then by assumption, there exists a semi*δ-open set U containing x such that s*δCl(U) ⊆B. Since x∈A, A∩U≠ϕ.This proves (iii).

(iii) ⟹(iv): Suppose A∩F=ϕ, where A is non-empty and F is semi*δ-closed. Then X\F is semi*δ-open and A∩ (X\F) ≠ϕ. By (iii), there exists a semi*δ-open set U such that A∩U≠ϕ, and U⊆s*δCl(U) ⊆X\F. Put V=X\s*δCl(U). Hence V is a semi*δ-open set containing B such that U∩V=U∩ (X\s*δCl(U)) ⊆U∩ (X\U)=ϕ. This proves (iv).

(iv) ⟹ (i). Let F be semi*δ-closed and x∉F. Take A={x}. Then A∩F=ϕ. By (iv), there exist disjoint semi*δ-open sets U and V such that U∩A≠ϕ and B⊆V. Since U∩A≠ϕ, x∈U. This proves that X is semi*δ-regular.

Theorem 3.3: If f is a semi*δ-irresolute and pre-semi*δ-closed injection of a topological space X into a semi*δ-regular space Y, then X is semi*δ-regular.

Proof: Let x∈X and U be a semi*δ-closed set in X not containing x. Since f is pre-semi*-closed, f(U) is a semi*δ-closed set in Y not containing f(x). Since Y is semi*δ-regular, there exist disjoint semi*δ-open sets V_1 and V_2 in Y such that f(x) ∈V_1 and f(U)⊆V_2. Since f is semi*δ-irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint semi*δ-open sets in X containing x and U respectively. Hence X is semi*δ-regular.

Theorem 3.4: If f is a semi*δ-continuous and closed injection of a topological space X into a regular space Y and if every semi*δ-closed set in X is closed, then X is semi*δ-regular.

Proof: Let x∈X and U be a semi*δ-closed set in X not containing x. Then by assumption, U is closed in X. Since f is closed, f(U) is a closed set in Y not containing f(x). Since Y is regular, there exist disjoint open sets V_1 and V_2 in Y such that f(x) ∈V_1 and f(U) ⊆V_2. Since f is semi*δ-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint semi*δ-open sets in X containing x and U respectively. Hence X is semi*δ-regular.

Theorem 3.5: If f : X⟶Y is a semi*δ-irresolute bijection which is pre-semi*δ-open and X is semi*δ-regular. Then Y is also semi*δ-regular.

Proof: Let f : X⟶Y be a semi*δ-irresolute bijection which is pre-semi*δ-open and X be semi*δ-regular. Let y∈Y and U be a semi*δ-closed set in Y not containing y. Since f is semi*δ-irresolute, by Theorem 2.7 $f^{-1}(U)$ is a semi*δ-closed set in X not containing $f^{-1}(y)$. Since X is semi*δ-regular, there exist disjoint semi*δ-open sets V_1 and V_2 containing $f^{-1}(y)$ and $f^{-1}(U)$ respectively. Since f is pre-semi*-open, f(V_1) and f(V_2) are disjoint semi*δ-open sets in Y containing y and U respectively. Hence Y is semi*δ-regular.

Theorem 3.6: If f is a continuous semi*δ-open bijection of a regular space X into a space Y and if every semi*δ-closed set in Y is closed, then Y is semi*δ-regular.

Proof: Let y∈Y and U be a semi*δ-closed set in Y not containing y. Then by assumption, U is closed in Y. Since f is a continuous bijection, $f^{-1}(U)$ is a closed set in X not containing the point $f^{-1}(y)$. Since X is regular, there exist disjoint open sets V_1 and V_2 in X such that
$f^{-1}(y)$∈V_1 and $f^{-1}(U)$⊆V_2. Since f is semi*δ-open, f(V_1) and f(V_2) are disjoint semi*δ-open sets in Y containing y and U respectively. Hence Y is semi*δ-regular.

Definition 3.7: A space X is said to be **s*δ-regular** if for every pair consisting of a point x and a closed set F not containing x, there are disjoint semi*δ-open sets U and V in X containing x and F respectively.

Theorem 3.8: For a topological space X, the following are equivalent:

(i) X is s*δ-regular.

(ii) For every x∈X and every open set U containing x, there exists a semi*δ-open set V containing x such that s*δCl(V) ⊆U.

(iii) For every set A and an open set B such that A∩B≠ϕ, there exists a semi*δ-open set U such that A∩U≠ϕ and s*δCl(U) ⊆B.

(iv) For every non-empty set A and closed set B such that A∩B=ϕ, there exist disjoint semi*δ-open sets U and V such that A∩U≠ϕ and B⊆V.

Proof: (i) ⟹(ii): Let U be an open set containing x. Then B=X\U is a closed set not containing x. Since X is s*δ-

regular, there exist disjoint semi*δ-open sets V and U containing x and B respectively. Then s*δCl(V) is disjoint from B, since if y ∈ B, the set U is a semi*δ-open set containing y disjoint from V. Hence s*δCl(V)⊆U.
(ii) ⟹(iii): Let A∩B≠ϕ and B be open. Let x∈A∩B. Then by assumption, there exists a semi*δ-open set U containing x such that s*δCl(U)⊆B. Since x∈A, A∩U≠ϕ.This proves (iii).
(iii) ⟹(iv): Suppose A∩B≠ϕ, where A is non-empty and B is closed. Then X\B is open and A∩(X\B) ≠ϕ. By (iii), there exists a semi*δ-open set U such that A∩U≠ϕ, and U⊆s*δCl(U) ⊆X\B. Put V=X\s*δCl(U). Hence V is a semi*δ-open set containing B such that U∩V=U∩(X\s*δCl(U)) ⊆U∩(X\U)=ϕ. This proves (iv).
(iv) ⟹(i). Let B be closed and x∉B. Take A={x}. Then A∩B=ϕ. By (iv), there exist disjoint semi*δ-open sets U and V such that U∩A≠ϕ and B⊆V. Since U∩A≠ϕ, x∈U. This proves that X is s*δ-regular.
Theorem 3.9: (i) Every s*δ-regular T_1 space is semi*δ-T_2.
(ii) Every semi*δ-regular semi*δ-T_1 space is semi*δ-T_2.
Proof: Suppose X is s*δ-regular and T_1. Let x and y be two distinct points in X. Since X is T_1, {x} is closed and y∉{x}. Since X is s*δ-regular, there exist disjoint semi*δ-open sets U and V in X containing {x} and y respectively. It follows that X is semi*δ-T_2. This proves (i). Suppose X is semi*δ-regular and semi*δ-T_1. Let x and y be two distinct points in X. Since X is semi*δ-T_1, {x} is semi*δ-closed and y∉{x}. Since X is semi*δ-regular, there exist disjoint semi*δ-open sets U and V in X containing {x} and y respectively. It follows that X is semi*δ-T_2. This proves (ii).

IV. NORMAL SPACES ASSOCIATED WITH SEMI*δ-OPEN SETS.

Definition 4.1: A space X is said to be **semi*δ-normal** if for every pair of disjoint semi*δ-closed sets A and B in X, there are disjoint semi*δ-open sets U and V in X containing A and B respectively.
Theorem 4.2: In a topological space X, the following are equivalent:
(i) X is semi*δ-normal.
(ii) For every semi*δ-closed set A in X and every semi*δ-open set U containing A, there exists a semi*δ-open set V containing A such that s*δCl(V) ⊆U.
(iii) For each pair of disjoint semi*δ-closed sets A and B in X, there exists a semi*δ-open set U containing A such that s*δCl(U)∩B=ϕ.
(iv) For each pair of disjoint semi*δ-closed sets A and B in X, there exist semi*δ-open sets U and V containing A and B respectively such that s*δCl(U)∩s*δCl(V)=ϕ.
Proof: (i) ⟹(ii): Let U be a semi*δ-open set containing the semi*δ-closed set A. Then B=X\U is a semi*δ-closed set disjoint from A. Since X is semi*δ-normal, there exist disjoint semi*δ-open sets V and W containing A and B respectively. Then s*δCl(V) is disjoint from B, since if y∈B, the set W is a semi*δ-open set containing y disjoint from V. Hence s*δCl(V)⊆U.
(ii) ⟹(iii): Let A and B be disjoint semi*δ-closed sets in X. Then X\B is a semi*δ-open set containing A. By (ii), there exists a semi*δ-open set U containing A such that s*δCl(U) ⊆X\B. Hence s*δCl(U)∩B=ϕ. This proves (iii).
(iii) ⟹(iv): Let A and B be disjoint semi*δ-closed sets in X. Then, by (iii), there exists a semi*δ-open set U containing A such that s*δCl(U)∩B=ϕ. Since s*δCl(U) is semi*δ-closed**,** B and s*δCl(U) are disjoint semi*δ-closed sets in X. Again by (iii), there exists a semi*δ-open set V containing B such that s*δCl(U)∩s*δCl(V)=ϕ. This proves (iv).
(iv) ⟹(i): Let A and B be the disjoint semi*δ-closed sets in X. By (iv), there exist semi*-δ open sets U and V containing A and B respectively such that s*δCl(U)∩s*δCl(V)=ϕ. Since U∩V⊆ s*δCl(U)∩s*δCl(V), U and V are disjoint semi*δ-open sets containing A and B respectively. Thus X is semi*δ-normal.
Definition 4.3: A space X is said to be **s*δ-normal** if for every pair of disjoint closed sets A and B in X, there are disjoint semi*δ-open sets U and V in X containing A and B respectively
Theorem 4.4: In a topological space X, the following are equivalent:
(i) X is s*δ-normal.
(ii) For every closed set F in X and every open set U containing F, there exists a semi*δ-open set V containing F such that s*δCl(V) ⊆U.
(iii) For each pair of disjoint closed sets A and B in X, there exists a semi*δ-open set U containing A such that s*δCl(U)∩B=ϕ.
Proof: (i) ⟹(ii): Let U be a open set containing the closed set F. Then H=X\U is a closed set disjoint from F. Since X is s*δ-normal, there exist disjoint semi*δ-open sets V and W containing F and H respectively. Then s*δCl(V) is disjoint from H, since if y∈H, the set W is a semi*δ-open set containing y disjoint from V. Hence s*δCl(V)⊆U.
(ii) ⟹(iii): Let A and B be disjoint closed sets in X. Then X\B is an open set containing A. By (ii), there exists a semi*δ-open set U containing A such that s*δCl(U) ⊆ X\B. Hence s*δCl(U)∩B=ϕ. This proves (iii).
(iii) ⟹(i): Let A and B be the disjoint semi*δ-closed sets in X. By (iii), there exists a semi*δ-open set U containing A such that s*δCl(U)∩B=ϕ. Take V=X\s*δCl(U). Then U and V are disjoint semi*δ-open sets containing A and B respectively. Thus X is s*δ-normal.

Theorem 4.5: If *f* is an injective and semi*δ-irresolute and pre-semi*δ-closed mapping of a topological space X into a semi*δ-normal space Y, then X is semi*δ-normal.

Proof: Let *f* be an injective and semi*δ-irresolute and pre-semi*δ-closed mapping of a topological space X into a semi*δ-normal space Y. Let A and B be disjoint semi*δ-closed sets in X. Since *f* is a pre-semi*δ-closed function, *f*(A) and *f*(B) are disjoint semi*δ-closed sets in Y. Since Y is semi*δ-normal, there exist disjoint semi*δ-open sets V_1 and V_2 in Y containing *f*(A) and *f*(B) respectively. Since *f* is semi*δ-irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint semi*δ-open sets in X containing A and B respectively. Hence X is semi*δ-normal.

Theorem 4.6: If f : X⟶Y is a semi*δ-irresolute surjection which is pre-semi*δ-open and X is semi*δ-normal, then Y is also semi*δ-normal.

Proof: Let f: X⟶Y be a semi*δ-irresolute surjection which is pre-semi*δ-open and X be semi*δ-normal. Let A and B be disjoint semi*δ-closed sets in Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint semi*δ-closed sets in X. Since X is semi*δ-normal, there exist disjoint semi*δ-open sets U_1 and U_2 containing $f^{-1}(A)$ and $f^{-1}(B)$ respectively. Since f is pre-semi*δ-open, $f(U_1)$ and $f(U_2)$ are disjoint semi*δ-open sets in Y containing A and B respectively. Hence Y is semi*δ-normal.

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