ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 4, April 2018

Strong (G, D) - Number of a Graph

[1] C. Santhaana Gomathi, [2] K. Palani, [3] S. Kalavathi
[1]PG Mathematics, [2]Department of Mathematics, [3] Research Scholar
[1][2]A.P.C. Mahalaxmi College for Women, Thoothukudi
[3]Manonmanium Sundaranar University, Tirunelveli

***Abstract:-*(G D) – number of a graph was introduced by Palani. K and Nagarajan. A. Let G be a (V, E) graph. A subset D of V (G) is said to be a (G, D)- set of G if it is both a dominating and a geodetic set of G. A dominating set is said to be a strong dominating set of G if it strongly dominates all the vertices of G. In this paper, we introduce the concept Strong (G,D)- number of a graph and find the same for some standard graphs and its bounds.**

I. INTRODUCTION

"Graph Theory" is an important branch of Mathematics. It has grown rapidly in recent times with a lot of research activities. In 1958, dominate on was formulized as a theoretical area in graph theory by C. Berge. He referred to the domination number as the coefficient of external stability and denoted as β(G). In 1962, Ore [6] was the first to use the term 'Domination' number by δ(G) and also he introduced the concept of minimal and minimum dominating set of vertices in graph. In 1977, Cockayne and Hedetniemi [5] introduced the accepted notation γ(G) to denote the domination number. Let G = (V, E) be any graph. A dominating set of a graph G is a set D of vertices of G such that every vertex in V-D is adjacent to atleast one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number of G. It is denoted by γ(G). The concept of geodominating (or geodetic) set was introduced by Buckley and Harary in [1] and Chartrand, Zhang and Harary in [2, 3, 4]. Let u, v ∈V(G). A u-v geodesic is a u-v path of length d (u, v). A vertex x is said to lie on a u-v geodesic p if x is any vertex on p. A set S of vertices of G is a geodominating (or geodetic) set if every vertex of G lies on an x-y geodesic for some x ,y in S. The minimum cardinality of geodominating set is the geodomination (or geodetic) number of G. It is denoted by g(G). K. Palani et.al [7,8,9] introduced the new concept (G,D)- set of graphs. A (G, D)- set of graph G is a subset S of vertices of G which is both dominating and geodominating (or geodetic) set of G. A (G,D)- set of G is said to bea minimal (G,D) set of G if no proper subset of S is a (G,D)- set of G. The minimum cardinality of all minimal (G, D)-set of G is called the (G,D)- number of G. It is denoted by $\gamma_G(G)$. Strong domination number was introduced by Sampathkumar and PushpaLatha [10]. A strong dominating set of a graph G is a set D⊆V(G) with the property that for all vertices x ϵ V(G)-D there is a vertex y ϵ N(x) in D with d(x) ≤ d(y) (i.e) every vertex not in D is dominated by a vertex in D having atleast the same degree. In this case we say that y strongly dominates x. The strong domination number γ_{st} (G) of a graph G is defined as the minimum cardinality of a strong dominating set of G. A 1-factor denoted by 1F is a regular spanning subgraph of degree 1.The generalized Hojos graph, denoted by $[K_n]$ is a graph having $n + nC_2$ vertices formed by joining each pair of vertices of K_n to vertex not in K_n. In this paper, we introduce the new concept strong (G, D)- number of a graph and proceed to find its bounds. Further, the strong (G,D)-number of some graphs are found.

The following results are from [7], [8] and [9].

1.1 Remark: Any (G,D)- set of G contains all its extreme vertices. In particular any (G,D)- set contains all its end vertices.

1.2 Theorem: $\gamma_G(P_n) = 2+\left\lceil\frac{n-4}{3}\right\rceil$.

1.3 Theorem: $\gamma_G(C_n) = \left\lceil\frac{n}{3}\right\rceil$

1.4 Theorem: $\gamma_G(K_n) = n$

II Main Results.

2.1 Definition:

A subset D of V(G) is said to be a (G,D)-set of G if it is both a dominating and geodetic set of G. A (G, D) set D is said to be a strong (G,D) set of G if for every vertex v ϵ V-D, there exists a vertex, u ϵ D such that d(u) ≥ d(v). The minimum cardinality of a strong (G, D)- set is called the strong (G,D)- number of G and is denoted by $s\gamma_G(G)$.

2.2 Observation :

1) Since any strong (G,D)- set is a (G,D)-set, by 1.1, any strong (G,D)-set contains all the extreme vertices of G.
2) $\gamma_G(G) \leq s\gamma_G(G)$.
3) In a regular graph G, all the vertices are of same degree. Therefore, any (G,D)- set is also a strong (G,D)- set.

Therefore $s\gamma_G(G) = \gamma_G(G)$ if G is regular.

4) By 3, $s\gamma_G(K_n) = \gamma_G(K_n) = n$

$s\gamma_G(C_n) = \gamma_G(C_n) = \left\lceil\frac{n}{3}\right\rceil$

2.3 Example: Consider the following graph G in figure 2.1

ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 4, April 2018

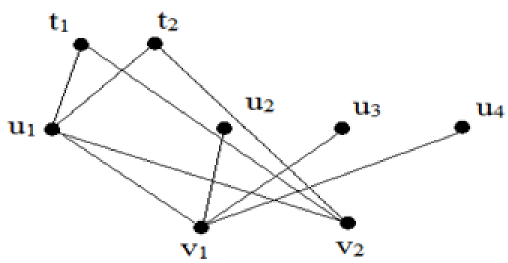


Figure 2.1:

Since u_2, u_3, u_4, t_1, t_2 are extreme vertices by observation 2.2 $\{u_2, u_3, u_4, t_1, t_2\}$ is contained in any strong (G,D)- set of G. Also, $\{u_2, u_3, u_4, t_1, t_2\}$---- (1) is a (G,D) set. But it is not a strong (G,D)- set. Since u_1, v_1, v_2 are not strong dominated by these vertices. Now to strong dominate u_1, v_1, v_2 add u_1 to (1).

Therefore, D= $\{u_1, u_2, u_3, u_4, t_1, t_2\}$ is a strong (G,D)-set of G. Further, it is a minimum strong (G,D)- set of G. Hence, $s\gamma_G = 6$.

2.4 Example:

Consider the Peterson graph in figure 2.2

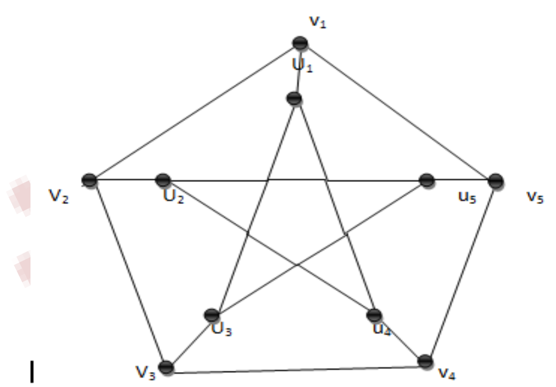


Figure 2.2 :

It is a 3-regular graph. Therefore, by 2.2, any (G,D)-set is a strong (G,D)-set. Since $d(u, v) \leq 2$ and between any two points there exists a unique u-v path of shortest length, we need a minimum of 3 from each of the 2 cycles in G to get a strong (G,D)- set. Therefore, $s\gamma_G(G) \geq 6$.

Also, $\{u_1, u_2, u_4, v_1, v_2, v_4\}$ is a strong (G,D)- set with 6 elements. Therefore, $s\gamma_G(G) \leq 6$.

Hence, $s\gamma_G(G) = 6$.

2.5 Theorem :

If G_1 and G_2 are two graphs, then $s\gamma_G(G_1 \cup G_2) = s\gamma_G(G_1) + s\gamma_G(G_2)$.

Proof: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$.

Let S_1, S_2 be minimum strong (G,D)- sets of G_1, G_2 respectively. Since $S_1 \cup S_2$ is a strong (G,D)-set of $G_1 \cup G_2$.

Therefore, $s\gamma_G(G_1 \cup G_2) \leq |S_1| + |S_2|$

$\leq s\gamma_G(G_1) + s\gamma_G(G_2)$.

Also, if D is a minimum strong (G,D)-set of $G_1 \cup G_2$, then obviously $D = D_1 \cup D_2$ where D_i is a strong (G,D)-set of G_i for i=1,2.

Since $s\gamma_G(G_1) \leq |D_1|$ and $s\gamma_G(G_2) \leq |D_2|$,

$s\gamma_G(G_1) + s\gamma_G(G_2) \leq |D_1| + |D_2|$

$= |D|$

$= s\gamma_G(G_1 \cup G_2)$ -(2)

By (1) & (2),

$s\gamma_G(G_1 \cup G_2) = s\gamma_G(G_1) + s\gamma_G(G_2)$

2.6 Remark:

In general $s\gamma_G(G_1 + G_2)$ need not be equal to $s\gamma_G(G_1) + s\gamma_G(G_2)$. For example, consider the following graphs in figure.

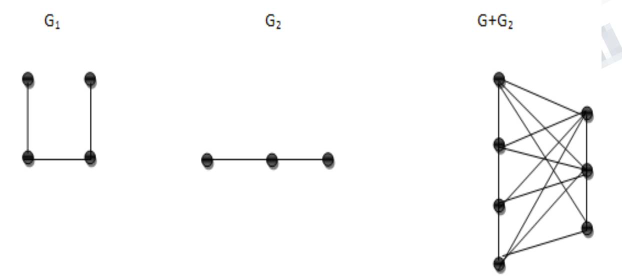


Figure 2.3

Here, $s\gamma_G(G_1) = 3$

$s\gamma_G(G_2) = 3$

$s\gamma_G(G_1 + G_2) = 4 \neq s\gamma_G(G_1) + s\gamma_G(G_2)$

2.7 Theorem:

$s\gamma_G(P_n) = 2 + \left\lceil \frac{n-2}{3} \right\rceil$ for $n \geq 2$

Proof:

Let $n \geq 2$ and $P_n = (v_1, v_2 \ldots\ldots v_n)$.

By observation 2.2, $\{v_1 v_n\} \subseteq$ any strong (G,D) set of P_n.

Let $S_1 = \{v_1, v_n\}$.

Obviously, S_1 along with any dominating set of $< V(P_n) - \{v_1, v_n\} >$ gives a strong (G,D) set of P_n.

Let S_2 be a minimum dominating set of $< V(P_n) - \{v_1, v_n\} >$

Clearly, $S_1 \cup S_2$ is a minimum strong (G,D) set of P_n and $< V(P_n) - \{v_1, v_n\} > = P_{n-2}$.

Therefore, $s\gamma_G(P_n) = |S_1 \cup S_2|$

$\leq |S_1| \cup |S_2|$ (since $S_1 \cap S_2 = \emptyset$)

$= 2 + \gamma(P_{n-2})$

$= 2 + \left\lceil \frac{n-2}{3} \right\rceil$

IFERP connecting engineers... developing research

ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 4, April 2018

2.8 Illustration:
Consider P_{13}
S = { $v_1, v_3, v_6, v_9, v_{12}, v_{13}$ } is a minimum strong (G,D) set of P_{13} and so $s\gamma_G$ (P_{13}) = |S| =6=2 + $\lceil \frac{11}{3} \rceil$

2.9 Theorem:
$s\gamma_G(K_{m,n})=$

$$\begin{cases} 2 \; if m = n = 1 \; or \; 3 \; if \; m = 1, n = 2 or \; m = 2, n = 1 \\ 4 \; if m > 2, n > 2 \; and m = n \\ m + 2 \; if \; m, n > 2 \; and n > m or \\ n + 2 \; if \; m, n > 2 \; and m > n \end{cases}$$

Proof:

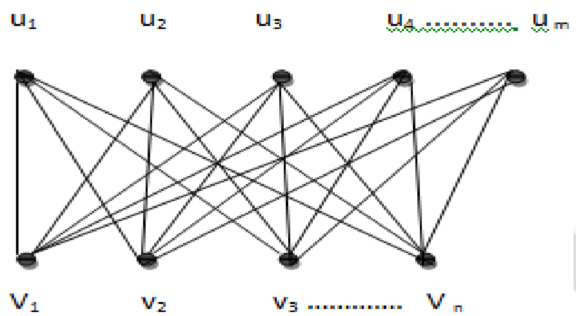


Figure 2.4

V ($K_{m,n}$) =V_1∪ V_2 where V_1={$u_1, u_2, \ldots u_m$} and V_2 ={$v_1, v_2 \ldots v_n$} is the bipartition of V

Case 1: Let m ; n ≤ 2
Subcase 1a: m=1, n=1
In this case, $K_{m,n} \simeq P_2$
Hence $s\gamma_G$ ($K_{m,n}$)= $s\gamma_G$ (P_2)=2
Subcase 1b: m = 1, n = 2 (or m = 2, n = 1)
Here, {u_1, v_1, v_2,} (or {v_1, u_1, u_2} forms a minimum strong (G,D) set of $K_{m,n}$.
Hence, $s\gamma_G$($K_{m,n}$) =32
Case 2: m >2 , n > 2 and m = n
In this case $K_{m,n}$ is regular.
Hence by observation 2.2,(2),
$s\gamma_G$ ($K_{m,n}$) = γ_G ($K_{m,n}$) =4
Case 3: m >2 , n > 2 and m ≠ n
Since m ≠ n, either m < n or n < m
W.L.O.G assume m < n
Obviously, S = {v_i ,v_j, u_k, u_l/ j ≤ m, 1 ≤ k, l ≤ n}
forms a minimum (G,D) –set. But, the vertices {u_1, u_2, …….u_{i-1}, u_{i+1},……u_{j-1}, u_{j+1}, ……..u_m } are not strong dominated by S. Further, the vertices of V_1 not adjacent to any vertex of degree greater than or equal to the degree of itself.
Hence, to strong dominate those vertices, any strong (G, D) must contain all the vertices of V_1.

Therefore, V_1∪ {v_i, v_j} form a minimum strong (G, D)-set of $K_{m,n}$ for all i, j such that i ≠ j and 1 ≤ i, j ≤ n
Hence, $s\gamma_G$($K_{m,n}$) = |V_1| + 2
= m + 2
Case 4: m >2 , n > 2 and m > n
The result follows the same lines of case 3..

2.10 Illustration:
Consider $K_{6,7}$graph.

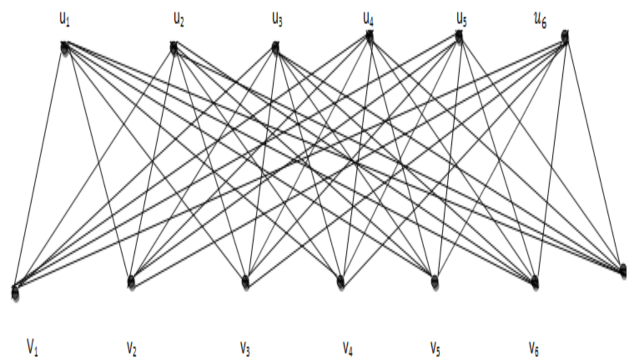


Figure 2.5

Here, {u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_5} is one of the minimum (G, D)set of G and so $\gamma_G(G)$ = 6+2 = 8.

2.11 Theorem:
Let G = (V, E) be any graph. S_1 be the set of extreme vertices of G and S_2 the set of all vertices of G with degree = ∆. Let S_3 denote the set of all isolated vertices of the subgraph induced by S_2. Then, $s\gamma_G(G)$≥ |S_1∪ S_3|.

Proof:
By observation 2.2, S_1 is a subset of any strong (G,D) set of G. Any vertex of degree ∆ can be strong dominated only by itself or any other vertex of same degree.
Therefore, if any vertex v of S_2 is an isolated vertex of <S_2>, then obviously v belongs to every strong (G, D)-set.
Therefore, S_1 ∪ S_3 is a subset of any minimum strong (G.D) set also.
Here, |S_1∪ S_3| ≤ $s\gamma_G(G)$.

2.12 Illustration:

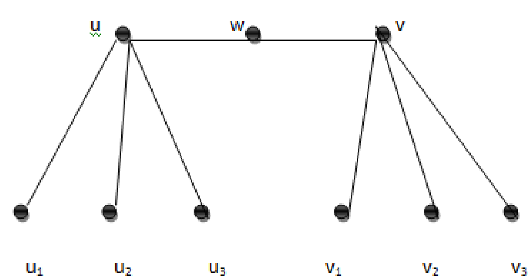


Figure 2.6

ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 4, April 2018

Here S = {u_1, u_2, u_3, v_1, v_2, v_3, w} is a (G, D)-set but not strong dominating. Since u and v are not strong dominated by any vertex of S.
Since u and v are vertex of degree ∆ which are non-adjacent, {u, v} ⊆ any strong (G ,D) set .
Here S^{l} = {u_1, u_2, u_3, v_1, v_2, v_3,u, v } is a minimum strong (G,D) set of G.
Hence, $s\gamma_G(G)$ =| S^{l}| = 8.

2.13 Theorem:
$s\gamma_G(K_{n,n}$ -1F) = 4.

Proof:

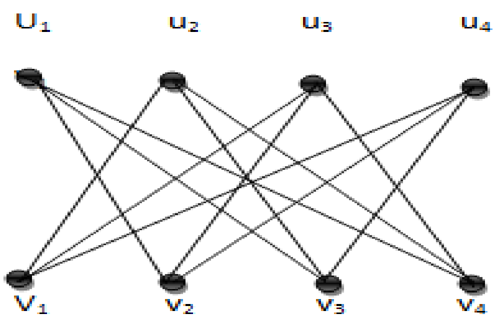


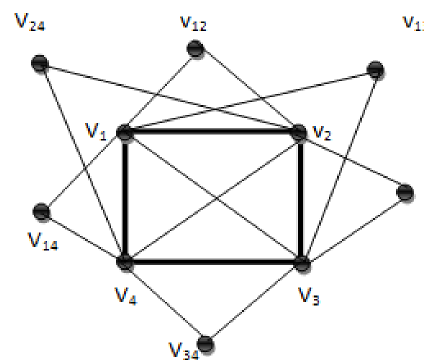
Figure 2.7

$K_{n,n}$ -1F is a bipartite graph with bipartition V_1, V_2 such that |V_1| =|V_2| = n.
Hence $K_{n,n}$ is a regular graph of degree n-1 {u_i, u_j/ 1 ≤ i , j ≤ n} ∪ { v_i, v_j / 1 ≤ i , j ≤ n} is a minimum (G,D) set. Further, $K_{m,n}$ –F is a regular. Hence every (G ,D)-set is also a strong (G,D) set.
Hence $s\gamma_G(K_{n,n}$ -1F) = 4.

2.14 Theorem:
Let [Kn] represent the generalized Hajos graph. Then, $\boldsymbol{s\gamma_G([K_n]) = \binom{n}{2} + 1}$.

Proof:
Label the vertices [K_n] as follows:
Let v_1, v_2,…… v_n represent the vertices of K_n and $v_{i,j}$ represent the vertex of (V([K_n]) - V(K_n)) which is adjacent to v_i and v_j of K_n.
It is observed that any geodesic connecting two vertices of K_n or two vertices outside K_n or one vertex of K_n and one vertex outside K_n contain only the vertices of K_n as internal vertices. Therefore, to get a geodetic covering we need to have all the outer vertices. But it is not strong dominating since the vertex in $K_n$are of degree ∆. Therefore the $\binom{n}{2}$ outer vertices together with one vertex of ofK_n forms a minimum strong (G, D)-set of [K_n].
Hence, $s\gamma_G([K_n]) = \binom{n}{2} + 1$

2.15 Illustration:



$[K_4] = \binom{4}{2} + 1 = 7$
Here S={$v_{i,j}$ / 1 ≤ i , j ≤ 4 and i ≠ j} is the set of vertices of K_4.. S∪{ v_i} is a minimum strong (G,D)- set of [K_4] for every i=1 to n4.
Hence, [K_4] = |S|+ 1 = 7.

BIBLIOGRAPHY

[1] Buckley F, Harary F and Quintas V L, Extremal results on the geodetic number of a graph, Scientia, volume A2 (1988), 17-26.

[2] Chartrand G, Harary F and Zhang P, Geodetic sets in graphs, DiscussionesMathematicae Graph theory, 20 (2000), 129-138e.

[3] Chartrand G, Harary F and Zhang P, On the Geodetic number of a graph, Networks, Volume 39(1) (2002), 1-6.

[4] Chartrand G, Zhang P and Harary F, Extremal problems in Geodetic graph Theory, CongressusNumerantium 131 (1998), 55-66.

[5]T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals Of Domination in graphs, Marcel Decker, Inc., New York 1998.

[6]Ore.O Theory of Graphs, American Mathematical Society Colloquium Publication 38 (American Mathematical Society Providence RI) 1962.

[7] Palani .K and Kalavathi. S, (G,D) – Number of some special graphs, International Journal Of Engineering and Mathematical Sciences January-June 2014, Volume 5,Issue-

1, pp.7-15ISSN(Print) – 2319 – 4537, (Online) – 2319 – 4545.

[8] Palani. K and Nagarajan. A (G,D) – number of graphs,International Journal Of Engineering and Mathematics Research. ISSN 0976 – 5840 Volume 3(2011), pp 285 -299.

[9] Palani. K ,Nagarajan. A and Mahadevan. G, Results connecting domination, geodetic and (G,D)– number of graph, International Journal Of Combinatorial graph theory and applications, Volume 3, No.1, January – June (2010)(pp.51 -59).

[10] E. Sampathkumar, L. PushpaLatha, Strong, weak domination and domination balance in a graph, Discrete Mathematics 161 (1996) 235–242.

