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A New Operational Matrices of Integration for Solving Integral Differential Equations of Fractional Order

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Abstract— A new techniques is presented in this paper to solve a class of fractional integral differential equations. By applying Riemann-Liouville's Properties with Bernstein polynomials, operational matrices of integration are introduced with collocation Chebyshev methods to obtain accurate solutions of the equations. Illustrative examples have shown that in some cases the proposed techniques yield the exact solutions.

Index Terms— Bernstein polynomials, Fractional Calculus.

I. INTRODUCTION

Bernstein matrices approach is the most important numerical methods to solve several applied mathematics problems. Recently, [1]-[3].

In this work, we introduce a class of fractional integro differential equation.

$$y^{\propto} = y(x) + \lambda_1 \int_0^x \frac{y^{\alpha}(t)}{\sqrt{x-t}} dt + \lambda_2 \int_0^x F(x,t) y^{\alpha}(t) dt + h(x) \qquad (1)$$

The initial conditions are

$$y^{\propto}(\delta) = y_i \qquad n-1 < \alpha \leq n \quad n \in N,$$
$$0 \leq \delta \leq R \qquad (2)$$

Where $F(x,t)$ and $h(x)$ both functions are continuous on the interval $[0, R]$, $y^{\alpha}(x)$ is the is the fractional derivative of $y(x)$, also δ, λ_i and y_i are constants.

In our paper, a new technique is presented to solve Eq. (1) by using the relations between the polynomials of Bernstein method $B_n(x)$ and their integration.

II. METHOD OF SOLUTIONS

By using the definition of Bernstein polynomials in [4, 5], we can write the approximate solution of Eq. (1) as

$$y_n(x) = B_n(x) C \qquad (3)$$

The approximate solution in (3) can be written as, see [5]

$$\mathrm{y(x)} = \mathrm{X(x)} D^T \mathrm{C} \qquad (4)$$

By using Riemann-Liouville fractional integral operator, we can write $J^{\alpha}[\mathrm{X(x)}]$ as

$$J^{\alpha}[\mathrm{X(x)}] = \left[\frac{\Gamma(1)}{\Gamma(\alpha+1)} x^{\alpha} \quad \frac{\Gamma(2)}{\Gamma(2+\alpha)} x^{\alpha+1} \quad \cdots \quad \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} x^{n+\alpha}\right] \qquad (5)$$

The relation (5) can be introduced as

$$J^{\alpha}[\mathrm{X(x)}] = [1 \quad \mathrm{x} \quad x^2 \ldots \mathrm{x}^n] \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(\alpha+1)} x^{\alpha} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} x^{n+\alpha} \end{bmatrix}$$

Then from (6) we have

$$J^{\alpha}[\mathrm{X(x)}] = \mathrm{X(x)}\Psi(\mathrm{x}), \qquad (6)$$

Where

$$\Psi(\mathrm{x}) = \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(\alpha+1)} x^{\alpha} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} x^{n+\alpha} \end{bmatrix} \qquad (7)$$

And

$$X(x) = [1 \quad \mathrm{x} \quad x^2 \ldots \mathrm{x}^n] \qquad (8)$$

We can define the integration of Eq. (4) as

$$J^{\alpha}[\mathrm{y(x)}] = J^{\alpha}[\mathrm{X(x)} D^T \mathrm{C}] = [J^{\alpha}\mathrm{X(x)}] D^T \mathrm{C} \qquad (9)$$

By using relations in (7) and (9), the operational matrix of integration J^{α} can be written as

$$J^{\alpha}[\mathrm{y(x)}] = \mathrm{X(x)}\Psi(\mathrm{x}) D^T \mathrm{C} \qquad (10)$$

For two parts $\lambda_1 \int_0^x \frac{y^{\alpha}(t)}{\sqrt{x-t}} dt$ *and* $\lambda_2 \int_0^x F(x,t) y^{\alpha}(t) dt$ please see [4]-[6]. By applying the operational matrix of integration J^{α} into (1) we find the relation

$$J^{\alpha}[y^{\propto}(x)] = J^{\alpha}[y(x)] + J^{\alpha}\left[\lambda_1 \int_0^x \frac{y^{\alpha}(t)}{\sqrt{x-t}} dt\right] + J^{\alpha}\left[\lambda_2 \int_0^x F(x,t) y \quad (t) dt\right]$$

$$+J^{\alpha}[h(x)] \quad (11)$$

Now substitute (10) into (11), we have

$$X(x)D^T C - X(x)\Psi(x)D^T C - \lambda_1 F_x D^T C - \lambda_2 S_x D^T C = y(0) + H(x), \quad (12)$$

By using the collocation points $\{xi : 0 \le i \le n\}$ (the roots of Chebyshev polynomials, see [2]) in Eq. (12) where

$$V = -X(x)\Psi(x)D^T + X(x)D^T - \lambda_1 K_x D^T - \lambda_2 S_x D^T. \quad (13)$$

So, the main matrix Eq. (12) corresponding to Eq. (1) can be formed as augmented matrix

$$VC = Z \quad (14)$$

By using the Gauss elimination method and remove all zero rows in matrix (18), we obtain a square matrix then the unknown matrix C is obtained as

$$C = V^{-1}Z \quad (15)$$

III. NUMERICAL RESULT AND DISCUSSION

Example 1.

Let's consider the fractional integro-DEs [7]

$$D^{0.15}y(x) = \frac{1}{4}\int_0^x \frac{y\ \ (t)}{\sqrt{x-t}}dt + \frac{1}{7}\int_0^x e^{x+t}y^{\alpha}(t)dt + g(x)$$

The initial condition is $y(0) = 0$ and

$$g(x) = \frac{\Gamma(3)}{\Gamma(2.85)}x^{1.85} - \frac{\Gamma(2)}{\Gamma(1.85)}x^{0.85} - \frac{\sqrt{\pi}\Gamma(3)}{4\Gamma\left(\frac{7}{2}\right)}x^{\frac{5}{2}} + \frac{\sqrt{\pi}\Gamma(2)}{2\Gamma\left(\frac{5}{2}\right)}x^{\frac{3}{2}} - \frac{e^{x+1} - 3e^{x}}{7}$$

The exact solution is

$$y(x) = x(1 - x).$$

By applying the techniques (I) in Section 2, with collocation points. For n=2 the approximate solution is

$$0.999999998x^2 - 0.999999998\,x$$

From figure 1 and Table 1, we can see the precise of results which obtained by Bernstein method. We compare the founded results with [7], can see our results are more accurate.

Table : 1

x	LWM [7]	LWCM [7]	Our method N=2	Exact
0	0	0	0	0
$\frac{2}{8}$	−0.1869	-0.1861	-0.1875	-0.1875
$\frac{4}{8}$	−0.2498	- 0.2497	-0.2500	-0.2500
$\frac{6}{8}$	−0.1869	-0.1862	-0.1875	-0.1875
$\frac{7}{8}$	−0.1084	-0.1081	-0.1093	-0.1084

Table 1: Comparison between approximate solutions and exact solution, with n = 2 and n = 3 for Example 1.

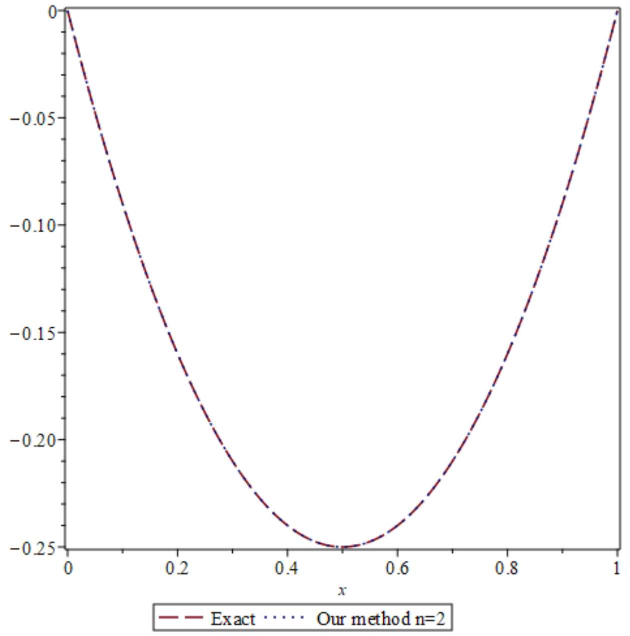

Figure 1: Comparison between approximate solutions and exact solution for Example 1, with $n = 2$.

IV. CONCLUSIONS

In this paper, new techniques based on operational matrices are presented to find results of integral differential equations of fractional order. The integral equations were converted to a linear system of equations. Collocation methods and Gauss elimination method are used to help us for finding the results more precisely.

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