ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 4, April 2018

Left Singularity and Left Regularity in near Idempotent Γ – Semi group

[1] M. Parvathi Banu, [2] D.Radha
[1] II M.Sc. Mathematics, [2] Assistant Professor in Mathematics
[1][2] A.P.C Mahalaxmi College for Women, Thoothukudi, India.

Abstract:- **In this paper, left singularity and left regularity in a near-idempotent Γ – semigroup are defined. In a near-idempotent Γ– semigroup λ_a is left singular and it is also proved that every δ class in a near-idempotent Γ – semigroup is left (right) singular if and only if S is left (right) regular. ξ- class is defined and proved that it is a near null semigroup. Also $\xi_a\ \xi_b \subset \xi$ for all a,b in S and $\xi_{ab}= \xi_a$ in a left singular near-idempotent Γ – semigroup. Any near-idempotent Γ – semigroup is left regular if and only if $\rho = \xi$ and right regular if $\lambda = \xi$. Also any near idempotent Γ – semigroup is near-commutative if $\delta = \xi$. Any near-commutative Γ – semigroup is near commutative if only and only if it is both left and right regular.**

Keywords:- Near- idempotent, Γ – semigroup, regular, singular semigroup, δ class , λ-class in near-idempotent Γ – semigroup, near-commutative Γ – semigroup.

1. INTRODUCTION

David Mclean[10] has obtained a decomposition of a band into more special bands. He has obtained a band as a semilattice union of rectangle bands. Motivated by this result, we have attempted to obtain a near idempotent Γ-semigroup as a union of more special near idempotent Γ-semigroups. We obtain each δ-class as a rectangular near-idempoent Γ- semigroup and each $\lambda(\rho)$ class as a left (right) singular near idempotent Γ-semigroups. We also show that a left(right) singular near idempotent Γ-semigroup is a semilattice union of left(right) singular near idempotent Γ-semigroups. We characterize left(right) regular Γ-semigroup in terms of the relations defined on it.

II. PRELIMINARY

DEFINITION II.1: Let S be a Γ- semigroup. Then S is said to be a near – idempotent Γ- semigroup if $x\gamma_1 y^2\gamma_2 z = x\gamma_1 y\gamma_2 z$ for all x,y,z ∈ S and $\gamma_1 , \gamma_2 \in \Gamma$

DEFINITION II.2: Let S be a Γ- semigroup. Then S is said to be left-regular near-idempotent Γ- semigroup if $x\gamma_1 y\gamma_2 z\gamma_3 y\gamma_4 w = x\gamma_1 y\gamma_2 z\gamma_3 w$ for all x,y,z,w ∈ S and $\gamma_1 , \gamma_2 ,\gamma_3, \gamma_4 \in \Gamma$

DEFINITION II.3: Let S be a Γ- semigroup. Then S is said to be left-singular near-idempotent Γ- semigroup if $x\gamma_1 y\gamma_2 z\gamma_3 w= x\gamma_1 y\gamma_2 w$ for all x,y,z,w∈ S and $\gamma_1 , \gamma_2 ,\gamma_3, \in \Gamma$

DEFINITION II.4: A semigroup R is called a rectangular near idempotent Γ −semigroup if R is a near idempotent semigroup and it satisfy the identity
$x\gamma_1 y\gamma_2 z\gamma_3 y\gamma_4 w= x\gamma_1 y\gamma_2 w$ for all x,y,z,w∈ S and $\gamma_1 , \gamma_2 ,\gamma_3, \gamma_4 \in \Gamma$

DEFINITION II.5: Let S be a near - idempotent Γ-semigroup and a and b, elements of S. We define the relation λ and ρ on S as follows:
a λ b if and only if $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1' a\gamma_2' y$ and $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1' b\gamma_2' y$ for all x, y ∈ S and $\gamma_1 , \gamma_2 ,\gamma_3, \gamma_1', \gamma_2' \in \Gamma$ a ρ b if and only if $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1' b\gamma_2' y$ and $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1' a\gamma_2' y$ for all x ,y∈ S and $\gamma_1, \gamma_2, \gamma_3, \gamma_1', \gamma_2' \in \Gamma$
Both λ and ρ turn out to be an equivalence relation on S.

LEMMA II.6: Let S be a near-idempotent Γ- semigroup. Then the relation λ is an equivalence relation on S.
Proof: $x\gamma_1 a^2\gamma_2 z = x\gamma_1 a\gamma_2 z$ for all x,y,a ∈ S and $\gamma_1 , \gamma_2 \in \Gamma$, by the definition of near-idempotent semigroup, so that a λ a for all a in S. Hence, λ is reflexive.

Let a λ b. Then, $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 a\gamma_2 y$ and $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1 b\gamma_2 y$ for all x,y ∈ S and $\gamma_1 , \gamma_2 ,\gamma_3 \in \Gamma$ which also implies b λ a. Hence, λ is symmetric.

Let a λ b and b λ c. Then, for all x, y ∈ S. We have, $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 a\gamma_2 y$ and $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1 b\gamma_2 y$ and $x\gamma_1 b\gamma_2 c\gamma_3 y = x\gamma_1 b\gamma_2 y$ and $x\gamma_1 c\gamma_2 b\gamma_3 y = x\gamma_1 c\gamma_2 y$. Hence $x\gamma_1 a\gamma_2 c\gamma_3 y = x\gamma_1\ a\gamma_2\ c\gamma_3 y = x\gamma_1\ a\gamma_2 b\gamma_3\ c\gamma_4 y = x\gamma_1\ a\gamma_2\ b\gamma_3 c\gamma_4\ y = x\gamma_1\ a\gamma_2 b\gamma_3\ y = x\gamma_1 a\gamma_2 y$ for all x,y ∈ S.

Similarly, $x\gamma_1 c\gamma_2 a\gamma_3 y = x\gamma_1\ c\gamma_2 b\gamma_3\ a\gamma_4\ y = x\gamma_1\ c\gamma_2\ b\gamma_3 a\gamma_4\ y = x\gamma_1\ c\gamma_2\ b\gamma_3\ y = x\gamma_1 c\gamma_2 y$ for all x, y ∈ S.

ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 4, April 2018

which implies a λ c. Hence λ is transitive. Thus λ is an equivalence relation on S.
Dually, we can prove that ρ is an equivalence relation on the near – idempotent Γ- semigroup on S.

LEMMA II.7: Let S be a near-idempotent Γ- semigroup. Let a λ b. Then, $a\gamma_1 c = b\gamma_2 c$ for all c ∈ S.

Proof: Let a λ b where a, b ∈ S. we claim that for any c ∈ S, $a\gamma_1 c = b\gamma_2 c$

a λ b ⇒ $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 a\gamma_2 y$ and $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1 b\gamma_2 y$ for all x, y ∈ S. Then for all x, y∈ S we have $x\gamma_1 a\gamma_2 c\gamma_3\ b\gamma_4 c\gamma_5 y = x\gamma_1 a\gamma_2\ c\gamma_3\ b\gamma_4 c\gamma_5\ y = x\gamma_1 a\gamma_2 b\gamma_3 c\gamma_4 b\gamma_5 c\gamma_6\ y = x\gamma_1\ a\gamma_2\ (b\gamma_3 c\gamma_4)^2\ y = x\gamma_1 a\gamma_2 b\gamma_3 c\gamma_4 y$ (by the definition of S) $= x\gamma_1 a\gamma_2 b\gamma_3\ c\gamma_4\ y = x\gamma_1 a\gamma_2 c\gamma_3 y$ and
$x\gamma_1\ b\gamma_2 c\gamma_3\ a\gamma_4 c\gamma_5\ y = x\gamma_1\ b\gamma_2\ c\gamma_3 a\gamma_4 c\gamma_5\ y = x\gamma_1\ b\gamma_2 a\gamma_3 c\gamma_4 a\gamma_5 c\gamma_6\ y = x\gamma_1 b\gamma_2\ (a\gamma_3 c\gamma_4)^2\ y = x\gamma_1\ b\gamma_2\, a\gamma_3 c\gamma_4\, y$(by the definition of S)$= x\gamma_1\ b\gamma_2 a\gamma_3\, c\gamma_4\, y = x\gamma_1 b\gamma_2 c\gamma_3 y$ leading to $a\gamma_1 c = b\gamma_2 c$ for all c ∈ S .Hence λ is a right congruence on S.
Dually, ρ is a left congruence on S.

RESULT II.8: We now consider the composition of two relations λ and ρ as follows

Let S be a near-idempotent Γ- semigroup. Then for any a, b ∈ S, we say that
a $\lambda \circ \rho$ b if there exists c ∈ S, such that a λ c and c ρ b

LEMMA II.9: If S is a near-idempotent Γ- semigroup, then $\lambda \circ \rho = \rho \circ \lambda$ in S.

Proof: we first prove that $\lambda \circ \rho \subset \rho \circ \lambda$. Let a $\lambda \circ \rho$b.Then there exists c ∈ S such that a λ c and c ρ b.
a λ c ⇒ $x\gamma_1 a\gamma_2 c\gamma_3 y = x\gamma_1 a\gamma_2 y$ and $x\gamma_1 c\gamma_2 a\gamma_3 y = x\gamma_1 c\gamma_2 y$ for all x,y∈ S ,$\gamma_1, \gamma_2, \gamma_3$, ∈ Γ. Choose d = $a\gamma_1 c\gamma_2 b$. Then for all x,y∈ S, $x\gamma_1 a\gamma_2 d\gamma_3 y = x\gamma_1\ a\gamma_2\ a\gamma_3 c\gamma_4 b\gamma_5\ y = x\gamma_1\ a^2\gamma_2\ c\gamma_3 b\gamma_4\ y = x\gamma_1\ a\gamma_2\ c\gamma_3 b\gamma_4\ y = x\gamma_1\ a\gamma_2 c\gamma_3 b\gamma_4\ y = x\gamma_1 d\gamma_2 y$ and
$x\gamma_1 d\gamma_2 a\gamma_3 y = x\gamma_1\ a\gamma_2\ c\gamma_3 b\gamma_4\ a\gamma_5 y = x\gamma_1\ a\gamma_2 b\gamma_3\ a\gamma_4 y = x\gamma_1 a\gamma_2 b\gamma_3\ a\gamma_4 c\gamma_5\ y = x\gamma_1 a\gamma_2 c\gamma_3 y$ (since b ρ c, ρ is a left congruence) $= x\gamma_1 a\gamma_2 y$.
But $x\gamma_1 a\gamma_2 c\gamma_3 y = x\gamma_1 a\gamma_2 y$. So that finally we get $x\gamma_1 d\gamma_2 a\gamma_3 y = x\gamma_1 a\gamma_2 y$. Therefore a ρ d .Similarly, $x\gamma_1 d\gamma_2 b\gamma_3 y = x\gamma_1 a\gamma_2 c\gamma_3 b\gamma_4 b\gamma_5 y = x\gamma_1 a\gamma_2 c\ \gamma_3 b^2\gamma_4 y = x\gamma_1 a\gamma_2 c\gamma_3 b\gamma_4 y = x\gamma_1 d\gamma_2 y$ for all x, y in S. $x\gamma_1 b\gamma_2 d\gamma_3 y = x\gamma_1\ b\gamma_2\ a\gamma_3 c\gamma_4 b\gamma_5\ y = x\gamma_1 b\gamma_2 a\gamma_3\ c\gamma_4 b\gamma_5\ y = x\gamma_1 b\gamma_2 a\gamma_3 b\gamma_4\ y = x\gamma_1\ b\gamma_2\ a\gamma_3 b\gamma_4 y = x\gamma_1\ c\gamma_2 b\gamma_3\ a\gamma_4 b\gamma_5 y = x\gamma_1 c\gamma_2 b\gamma_3 y$ (since λ is a right congruence $a\gamma_1 b$ λ $c\gamma_2 b$). But $x\gamma_1 c\gamma_2 b\gamma_3 y = x\gamma_1 b\gamma_2 y$, so that we get $x\gamma_1 b\gamma_2 d\gamma_3 y = x\gamma_1 b\gamma_2 y$. Hence d λ c. Thus a ρ d , d λ b so that a $\rho \circ \lambda$ b. This gives $\lambda \circ \rho \subset \rho \circ \lambda$. By similar argument, we can prove that
$\rho \circ \lambda \subset \lambda \circ \rho$. Thus we get $\lambda \circ \rho = \rho \circ \lambda$.
We now define the relation δ on S as follows

DEFINITION II.10: Let S be a near –idempotent Γ- semigroup. Let a, b ∈ S. We define $\delta = \lambda \circ \rho$. In other words, a δ b if and only if there exists c ∈ S such that a λ c and c ρ b
We have already prove that $\lambda \circ \rho = \rho \circ \lambda$. Hence we can write a $\lambda \circ \rho$ b or a $\rho \circ \lambda$ b instead for a δ b.

LEMMA II.11: Let S be a near-idempotent Γ- semigroup. δ is an equivalence relation on S.

Proof: For all a in S, a λ a and a ρ a. Since λ and ρ are reflexive so that a $\lambda \circ \rho$ b which means a δ a. Hence δ is reflexive.

a δ b ⇒ a $\lambda \circ \rho$ b ⇒ there exists u ∈ S such that a λ u and u ρ b ⇒ there exists u ∈ S such that b ρ u and u λ a since λ and ρ are symmetric ⇒ b $\rho \circ \lambda$ a ⇒b δ *a* [since $\lambda \circ \rho = \rho \circ \lambda = \delta$]. Hence δ is symmetric.

a δ b , b δ c ⇒ there exists u, v ∈ S such that a λ u and u ρ b , b λ v and v ρ c since u ρ b and b λ v we have u $\rho \circ \lambda$ v we have u $\lambda \circ \rho$ v . Since $\lambda \circ \rho = \rho \circ \lambda$. Thus there exists w ∈ S such that u λ w and w ρ v.
a λ u and u λ w so that a λ w ; w ρ v and v ρ c so that w ρ c. Therefore a $\lambda \circ \rho$ c
i.e., a δ c. Thus δ is transitive. Hence δ is an equivalence relation on S.

III. DECOMPOSITION OF NEAR IDEMPOTENT Γ-SEMIGROUP

Theorem III.1 : { δ_a / a ∈S } is a semigroup under the operation $\delta_a * \delta_b = \delta_{ab}$ We now prove that every δ- class is a Γ- subsemigroup of S.

Theorem III.2: Let S be a near-idempotent Γ-semigroup and a ∈ S.Then δ_a is rectangular near-idempotent Γ-semigroup.

Proof: Let x ,y , z, w ∈ δ_a . x δ a, y δ a, z δ a , w δ a. By transitivity y δ z. Hence for all x, w ∈ S. $x\gamma_1 y\gamma_2 z\gamma_3 y\gamma_4 w = x\gamma_1 y\gamma_2 w$ and $x\gamma_1 z\gamma_2 y\gamma_3 z\gamma_4 w = x\gamma_1 z\gamma_2 w$. This result is true when x, w ∈ δ_a also. Thus we have $x\gamma_1 y\gamma_2 z\gamma_3 y\gamma_4 w = x\gamma_1 y\gamma_2 w$ for all x,y,z,w ∈ δ_a .Hence δ_a is rectangular near idempotent Γ-semigroup.

Theorem III.3: Let S be a near-idempotent Γ- semigroup. Then for a∈ S, λ_a is left-singular near idempotent Γ-semigroup.

Proof: Let S be a near-idempotent Γ- semigroup. Consider the relation λ on S. For a, b ∈ S a λ b if and only if $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 a\gamma_2 y$ and $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1 b\gamma_2 y$ for all $x, y \in S$ and $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ .Consider an equivalence relation λ_a where $a \in S$.We claim that λ_a is a near left – singular near-idempotent Γ-semigroup. Let $u, v \in \lambda_a$. a λ u and a λ v . For all $x, y \in S$ $x\gamma_1 a\gamma_2 u\gamma_3 y = x\gamma_1 a\gamma_2 y$, $x\gamma_1 u\gamma_2 a\gamma_3 y = x\gamma_1 u\gamma_2 y$ and $x\gamma_1 a\gamma_2 v\gamma_3 y = x\gamma_1 a\gamma_2 y$, $x\gamma_1 v\gamma_2 a\gamma_3 y = x\gamma_1 a\gamma_2 y$. For all x , $y \in S$. $x\gamma_1\ u\gamma_2 v\gamma_3\ a\gamma_4 y = x\gamma_1 u\gamma_2 . v\gamma_3 a\gamma_4\ y = x\gamma_1 u\gamma_2 v\gamma_3 y$ and $x\gamma_1 a\gamma_2\ u\gamma_3 v\gamma_4\ y = x\gamma_1 a\gamma_2 u\gamma_3 \cdot v\gamma_4\ y = x\gamma_1\ a\gamma_2 v\gamma_3\ y = x\gamma_1 a\gamma_2 y$. Here, $u\gamma v\ \lambda$ a. Hence $u\gamma v \in \lambda_a$.

λ_a is a subsemigroup of S. Also $x\gamma_1 u\gamma_2 v\gamma_3 = x\gamma_1\ u\gamma_2 a\gamma_3\ v\gamma_4 y = x\gamma_1 u\gamma_2\ \ a\gamma_3 v\gamma_4\ y = x\gamma_1 u\gamma_2\ \ a\gamma_3\ \ y = x\gamma_1 u\gamma_2 y$ and $x\gamma_1 v\gamma_2 u\gamma_3\ y = x\gamma_1\ v\gamma_2 a\gamma_3\ \ u\gamma_4 y = x\gamma_1 v\gamma_2\ \ a\gamma_3 u\gamma_4\ y = x\gamma_1 v\gamma_2 a\gamma_3 y = x\gamma_1\ v\gamma_2 y$ for all x, y in S. Hence it is also true for all $x, y \in \lambda_a$. Thus for

$x, u, v, y \in \lambda_a$, $x\gamma_1\ u\gamma_2 v\gamma_3 y = x\gamma_1 u\gamma_2 y$. Hence λ_a is a left – singular near-idempotent Γ- semigroup.

Theorem III.4: Let R be a rectangular near-idempotent Γ-semigroup. Then for $a, b \in R$, $\lambda_a \lambda_b \subset \lambda_b$

Proof: Let $u \in \lambda_a$ and $v \in \lambda_b$ .Then $x\gamma_1 u\gamma_2 a\gamma_3 y = x\gamma_1 u\gamma_2 y$ and $x\gamma_1 a\gamma_2 u\gamma_3 y = x\gamma_1 a\gamma_2 y$, $x\gamma_1 v\gamma_2 b\gamma_3 y = x\gamma_1 v\gamma_2 y$ and $x\gamma_1 b\gamma_2 v\gamma_3 y = x\gamma_1 b\gamma_2 y$.

$x\gamma_1 u\gamma_2 v\gamma_3\ \ b\gamma_4\ y = x\gamma_1 u\gamma_2\ \ v\gamma_3 b\gamma_4\ y = x\gamma_1 u\gamma_2 v\gamma_3 y$ and $x\gamma_1 b\gamma_2\ \ u\gamma_3 v\gamma_4 y = x\gamma_1 b\gamma_2 v\gamma_3\ \ u\gamma_4 v\gamma_5\ \ y = x\gamma_1 b\gamma_2\ v\gamma_3 u\gamma_4 v\gamma_5 y = x\gamma_1\ \ b\gamma_2 v\gamma_3\ \ y = x\gamma_1 b\gamma_2 y$ [since $u, v \in R$ and hence $x\gamma_1 v\gamma_2 u\gamma_3 v\gamma_4 y = x\gamma_1 v\gamma_2 y$]. Thus, $u\gamma v \in \lambda_b$ i.e, $\lambda_a \lambda_b \subset \lambda_b$ for all a ,b in R

Note III.5: If we define an operation $\diamond$ on $\{ \lambda_a / a \in R \}$ such that $\lambda_a \diamond \lambda_b = \lambda_c$ if and only if $\lambda_a \lambda_b \subset \lambda_c$ then from the above discussions of this theorem it is clear that $\lambda_a \diamond \lambda_b = \lambda_b$. Thus R is right-singular band of left – singular near-idempotent Γ-semigroup.

Now we move on to verify that left (right) regular near-idempotent Γ- semigroup is a semilattice of left (right) singular near-idempotent Γ- semigroup.

Theorem III.6: S is a left (right) regular near-idempotent Γ-semigroup if and only if every δ-class in S is a near left (right) singular near-idempotent Γ- semigroup.

Proof: Let S be a left (right)regular near-idempotent Γ-semigroup. Then $x\gamma_1 y\gamma_2 z\gamma_3 y\gamma_4 w = x\gamma_1 y\gamma_2 z\gamma_3 w$ for all x, y, z, $w \in S$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$ ---------(1). Let $a \in S$. δ_a is a rectangular near-idempotent Γ-semigroup of S. Hence, $x\gamma_1 y\gamma_2 z\gamma_3 y\gamma_4 w = x\gamma_1 y\gamma_2 w$ for all x, y, w, z in δ_a -----------(2)

(1) and (2) gives $x\gamma_1 y\gamma_2 z\gamma_3 w = x\gamma_1 y\gamma_2 w$ for all x,y, z, w in δ_a . Hence δ_a degenerates into a near left singular near-idempotent Γ-semigroup .

Conversely, let $a, b \in S$. $a\gamma_1 b\ \delta\ b\gamma_2 a$, ab, ba are in the same δ-class. They are in a near- idempotent Γ-semigroup. For all x, $y \in S$. $x\gamma_1\ a\gamma_2 b\gamma_3\ \ \ b\gamma_4 a\ \gamma_5\ \ y = x\gamma_1 a\gamma_2 b\gamma_3 y \Rightarrow x\gamma_1\ a\gamma_2 b^2\gamma_3 a\gamma_4 y = x\gamma_1 a\gamma_2 b\gamma_3 y \Rightarrow x\gamma_1\ a\gamma_2 b\gamma_3\ a\ \gamma_4 y = x\gamma_1 a\gamma_2 b\gamma_3 y$.

Therefore S is a left –regular near idempotent Γ-semigroup.

IV. LEFT SINGULARITY AND LEFT REGULARITY IN NEAR IDEMPOTENT Γ- SEMIGROUP

DEFINITION IV.1: Let S be a near-idempotent Γ-semigroup. Let $a, b \in S$.We say that a ξ b if and only if a λ b and a ρ b. In other words, $\xi = \lambda \cap \rho$.

LEMMA IV.2: Let S be a near-idempotent Γ-semigroup. Let $a, b \in S$. Then a ξ b if and only if $x\gamma_1 a\gamma_2 y = x\gamma_1 b\gamma_2 y$ for all $x, y \in S$.

Proof: let a ξ b. Then a λ b and a ρ b. Hence for all $x, y \in S$. $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 a\gamma_2 y$; $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1 b\gamma_2 y$ and $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 b\gamma_2 y$; $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1 a\gamma_2 y$. From the above equation it is clear that, $x\gamma_1 a\gamma_2 y = x\gamma_1 b\gamma_2 y$ for all x,y in S. Conversely, suppose that $x\gamma_1 a\gamma_2 y = x\gamma_1 b\gamma_2 y$ for all x,y $\in S$. For all x,y $\in S$. $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 b\gamma_2 b\gamma_3 y = x\gamma_1 b^2\gamma_2 y = x\gamma_1 b\gamma_2 y$ and $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1 a\gamma_2 a\gamma_3 y = x\gamma_1 a^2\gamma_2 y = x\gamma_1 a\gamma_2 y$ so that a ρ b. Also for all $x, y \in S$. $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 a\gamma_2 a\gamma_3 y = x\gamma_1 a^2\gamma_2 y = x\gamma_1 a\gamma_2 y$ and $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1 b\gamma_2 b\gamma_3 y = x\gamma_1 b^2\gamma_2 y = x\gamma_1 b\gamma_2 y$. So that, a λ b.Thus a $(\lambda \cap \rho)$ i.e a ξ b.

LEMMA IV.3: Let S be a near-idempotent Γ-semigroup. Let $a \in S$, then every ξ- class is a near null semigroup.

Proof: Define ξ on S. Let $a \in S$. Let $u, v \in \xi_a$. $x\gamma_1 u\gamma_2 y = x\gamma_1 a\gamma_2 y = x\gamma_1 v\gamma_2 y$ for all x , $y \in S$. For all $x, y \in S$. $x\gamma_1 u\gamma_2 v\gamma_3 y = x\gamma_1\ \ u\gamma_2 .\ v\gamma_3 y = x\gamma_1 a\gamma_2 v\gamma_3 y = x\gamma_1 a\gamma_2 .v\ \gamma_3 y = x\gamma_1 a\gamma_2 .\ a\gamma_3 y = x\gamma_1\ \ a^2\gamma_2 y = x\gamma_1 a\gamma_2 y$. Then $u\gamma v \in \xi_a$ so that ξ_a is subsemigroup of S. Also $x\gamma_1 u\gamma_2 y = x\gamma_1 v\gamma_2 y$ for all x, $y \in S$. Hence $x\gamma_1 u\gamma_2 y = x\gamma_1 v\gamma_2 y$ for all $x, y \in \xi_a$ also. In other words if $x, y, z, w \in \xi_a$. $x\gamma_1 y\gamma_2 w = x\gamma_1 z\gamma_2 w$. Hence ξ_a is a near null semigroup. Also, if $u \in \xi_a$ and $v \in \xi_a$. For all x, y in S, $x\gamma_1 u\gamma_2 y = x\gamma_1 a\gamma_2 y$ and $x\gamma_1 v\gamma_2 y = x\gamma_1 b\gamma_2 y$. $x\gamma_1 u\gamma_2 v\gamma_3 y = x\gamma_1 a\gamma_2 v\gamma_3 y = x\gamma_1 a\gamma_2 b\gamma_3 y$. So that $u\gamma v \in \xi_{ab}$. Hence $\xi_a \xi_b \subset \xi_{ab}$.

LEMMA IV.4: Let S be a near-idempotent Γ-semigroup and $a, b \in S$. Then $\xi_a \xi_b \subset \xi_{ab}$.

LEMMA IV.5: Let $\Xi = \{ \xi_a / a \in S \}$. Define ▫ on Ξ such that ξ_a ▫ $\xi_b = \xi_c$ if and only if $\xi_a \xi_b \subset \xi_c$. Then Ξ is a semigroup under ▫ .

Proof: By the last lemma $\xi_a \xi_b \subset \xi_{ab}$. Hence ξ_a ▫ $\xi_b = \xi_{ab}$. Hence Ξ is a semigroup under ▫ .

LEMMA IV.5: Let L be a left singular near-idempotent Γ-semigroup a, b in L. Then $\xi_{ab} = \xi_a$.

Proof: Let $x, y, a, b \in L$. Then $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 a\gamma_2 y$ for all $x, y \in L$. Hence $\xi_{ab} = \xi_a$. Thus ξ_a ▫ $\xi_b = \xi_{ab} = \xi_a$ for all a, b ∈

L. Hence a left singular near-idempotent Γ-semigroup is a left singular union of near null semigroups.

LEMMA IV.6: A right singular near-idempotent Γ-semigroup is a right singular union of near null semigroups.

LEMMA IV.7: A near-idempotent Γ-semigroup S is a left regular near-idempotent Γ-semigroup if and only if $\lambda = \delta$ on S.

LEMMA IV.8: A near-idempotent Γ-semigroup S is a left regular near-idempotent Γ-semigroup if and only if $\rho = \xi$ on S.

Proof: In a left regular near-idempotent Γ-semigroup $\xi = \lambda \cap \rho = \delta \cap \rho$ [by last lemma] $= \rho$ since $\rho \subset \delta$. Let S be a near-idempotent Γ-semigroup in which $\rho = \xi$. $x\gamma_1 u\gamma_2 v\gamma_3 u\gamma_4 . u\gamma_5 v\gamma_6 \; y = x\gamma_1 \; u\gamma_2 v\gamma_3 \; u^2\gamma_4 v\gamma_5 y = x\gamma_1 u\gamma_2 v\gamma_3 u\gamma_4 v\gamma_5 y = x\gamma_1(u\gamma_2 v\gamma_3)^2 y$ for all x, y, u, v ∈ S $= x\gamma_1 u\gamma_2 v\gamma_3 y$.

$x\gamma_1 \; u\gamma_2 v\gamma_3 . u\gamma_4 u\gamma_5 u\gamma_6 \; y = x\gamma_1(u\gamma_2 v\gamma_3)^2 u\gamma_4 y = x\gamma_1 \; u\gamma_2 v\gamma_3 u\gamma_4 y$. Thus $u\gamma_1 v \; \rho \; u\gamma_2 v\gamma_3 u$. Since $\rho = \xi$, $x\gamma_1 u\gamma_2 v\gamma_3 u\gamma_4 \; y = x\gamma_1 u\gamma_2 v\gamma_3 y$ for all u, v ∈ S. Hence S is left regular near-idempotent Γ-semigroup.

Lemma IV.9: λ is a congruence relation in a left regular near-idempotent Γ-semigroup S.

Proof: Let S be a left regular near-idempotent Γ-semigroup. Let a λ b. Then $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 a\gamma_2 y$; $x\gamma_1 b\gamma_2 a\gamma_3 y = x\gamma_1 b\gamma_2 y$; Let c ∈ S. $x\gamma_1 c\gamma_2 a\gamma_3 c\gamma_4 b\gamma_5 y = x\gamma_1 c\gamma_2 a\gamma_3 c\gamma_4 . b\gamma_5 y = x\gamma_1 c\gamma_2 a\gamma_3 b\gamma_4 y = x\gamma_1 c\gamma_2 a\gamma_3 y$.

$x\gamma_1 c\gamma_2 b\gamma_3 c\gamma_4 a\gamma_5 y = x\gamma_1 c\gamma_2 b\gamma_3 a\gamma_4 y = x\gamma_1 c\gamma_2 b\gamma_3 y$. Thus we get that a λ b$\Rightarrow$ $c\gamma_1 a \; \lambda \; c\gamma_2 b$. Therefore λ is a left congruence. We know that λ is a right congruence in a near-idempotent Γ-semigroup. Thus λ is a congruence relation on S.

Lemma IV.10: In a near-idempotent Γ-semigroup S, $\delta = \xi$ implies that S is a near-commutative near idempotent Γ-semigroup.

Proof: Let a, b ∈ S. In any near– idempotent Γ-semigroup $a\gamma_1 b \; \lambda \; b\gamma_2 a$. But $\delta = \xi$. Hence $a\gamma_1 b \; \xi \; b\gamma_2 a$. Thus $x\gamma_1 a\gamma_2 b\gamma_3 y = x\gamma_1 b\gamma_2 a\gamma_3 y$ for all x, y in S. Hence S is near-commutative.

Theorem IV.11: A near-idempotent Γ-semigroup S is a near-commutative if and only if $\delta = \xi$ on S.

Theorem IV.12: A near-idempotent Γ-semigroup S is near-commutative if and only if it is both a left regular and a right regular near-idempotent Γ-semigroup.

Proof: Suppose that near-idempotent Γ-semigroup S is a near-commutative near-idempotent Γ-semigroup. Then $x\gamma_1 y\gamma_2 z\gamma_3 w = x\gamma_1 z\gamma_2 y\gamma_3 w$ for all x, y, z, w in S.

$x\gamma_1 y\gamma_2 z\gamma_3 y\gamma_4 w = x\gamma_1 \; y\gamma_2 z\gamma_3 . \; y\gamma_4 \; w = x\gamma_1 \; y^2\gamma_2 \; z\gamma_3 w = x\gamma_1 \, y\gamma_2 z\gamma_3 w$. Therefore S is a left regular near – idempotent Γ-semigroup. $x\gamma_1 y\gamma_2 z\gamma_3 y\gamma_4 w = x\gamma_1 y\gamma_2 . z\gamma_3 y\gamma_4 \, w = x\gamma_1 z\gamma_2 y^2 \gamma_3 w = x\gamma_1 z\gamma_2 y \gamma_3 w$

Therefore S is a right regular near-idempotent Γ-semigroup. Therefore S is both a left regular and a right regular near-idempotent Γ-semigroup.

Conversely, Let S be both a left regular and a right regular near-idempotent Γ-semigroup. $x\gamma_1 y\gamma_2 z\gamma_3 \quad y\gamma_4 w = x\gamma_1 y\gamma_2 z\gamma_3 w$ by near left regularity $x\gamma_1 y\gamma_2 z\gamma_3 y\gamma_4 w = x\gamma_1 z\gamma_2 y\gamma_3 w$ by near right regularity. Therefore $x\gamma_1 y\gamma_2 z\gamma_3 w = x\gamma_1 z\gamma_2 y\gamma_3 w$. So that S is near-commutative.

Conclusion: In this paper, the class δ_a is proved as a rectangular near-idempotent Γ-semigroup and the class λ_a is proved as a left singular near-idempotent Γ-semigroup and for any a, b in a rectangular near-idempotent Γ-semigroup, $\lambda_a \lambda_b$ is contained in λ_b. Also, R is a right singular band of left singular near-idempotent Γ-semigroup. Also a relation ξ is defined and is proved that $\xi = \lambda \cap \rho$ along with the property that $\xi_a \xi_b \subseteq \xi_{ab}$ for any a, b in S. Also, if S is left-singular then $\xi_a \xi_b = \xi_a$.

REFERENCES

1. Clifford, A.H. and Preston, G.B., " *The Algebraic Theory of Semigroups*", Vol. I,Providence, 1961

2. A. H. Clifford, *Bands of semigroups*, Proc. Amer. Math. Soc. 5 (1954), 499-504

3. Jayalakshmi, A., " *Some Studies in Algebraic Semigroups and Associated Structures",* Doctoral Thesis, Bangalore University, 1981.

4. T.N. Kavitha, A.Jayalakshmi, *On near – idempotent semigroups*, Vol 5, No 5, International Journal of Mathematical Archive May 2014.

5. T.N. Kavitha, A. Jayalakshmi, *Rectangular near-idempotent semigroup*, -selected for presentation and publication at the international conference on mathematical science- 2014 and will be published in mathematical science international research journal, Vol 3, issue 2.

6. T.N. Kavitha, A. Jayalakshmi, *Some Special Types Of Near idempotent Semigroup* – selected for presentation and publication at the IMRF international conference – 2014, Thailand and will be published in

7. Kimura, N., *The Structure of Idempotent Semigroups* (I), Pacific J. of Math.(1958)

8. Naoki Kimura, *Note on idempotent semigroups.* II, Proc. Japan. Acad. 34 (1958), 110-112 .

International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 4, April 2018

9. Naoki Kimura, *On semigroups*, Dissertation, Tulane University (1957), 1-133.

10. Mc Lean, D., *Idempotent Semigroup*, Am. Math. Mon. 61 (1954)

11. Yamada. M, *on the greatest semilattice decomposition of a semigroup*, Kodai math. Sem. Rep.7 (1955) (59-62 Mathematical Science International Research Journal- Vol 3, special issue.

12 . Jayalakshmi, A., and. Atre. Ananth K. " *On Generalization of the concept of Inflation of semigroups*" Academic Journal of Sciences", CD. ROM. ISSN 2165 – 6282:: 2(2): 527- 532(2013).