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Mathematical Modelling of Population Growth

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Abstract:-- **We cannot have a sustainable planet without stabilizing population. As human population increase, humans demand for resources like water, land, trees, and energy. Unfortunately, the price of all this "increase and demand" is paid for by the other endangered plants, animals and natural resources in an increasingly volatile and dangerous climate. This necessitates a mathematical model to predict the future population in terms of growth rate and population figures with reasonably virtuous accuracy. Mathematics being one of the languages of sciences, Mathematical models can predict the behaviour of systems based on physics, chemistry, biology etc. There are certain mathematical models to effectively predict economic and social systems including the population growth. The present work deals with mathematical modelling of population growth using exponential and logistic growth model, which is nothing but the differential equations, with which we can study the changes in size of populations through time, which helps us predict the population of a certain place at a certain time. The prediction is compared with the actual population of the past, based on the model which predicts the population with better accuracy, which can be used to predict the growth rate of the future population.**

keywords:-- **Mathematical modelling, Population growth, Logistic growth, Exponential growth, Growth rate, Differential equations.**

INTRODUCTION

Projection of any country's population plays a significant role in the planning as well as in the decision making for the socio-economic and demographic development. Today the major issue of the world is the tremendous growth of the population. Any truly meaningful conservation and sustainability efforts must take the expanding human population footprint into consideration. Globally, over several thousands of people are added every day — each need sufficient land, water, shelter, food, and energy for a decent life. In order to provide better living condition to the people in terms of basic needs like food, water, education, health care etc., the society requires to plan and execute appropriate schemes in time bound manner. Mathematical Modelling is a broad interdisciplinary science that uses mathematical and computational techniques to model and elucidate the phenomena arising in life sciences which is created in the hope that the behaviour it predicts will resemble the real behaviour on which it is based. It involves the following processes.

1) The mimicking of a real-world problem in mathematical terms: thus, the construction of mathematical model.
2) The analysis or solution of the resulting mathematical problem.
3) The interpretation of the mathematical results in the context of the original situation.

A model can be in many shapes, sizes and styles. It is important to highlight that a model is not reality but merely a human construct (representing the reality as much as possible) to help us better understand real-world system. One uses models in order to extract the important trend from complex processes to permit comparison among systems to facilitate analysis of causes of processes acting on the system and to make a prediction about the future.

At first glance, modelling the growth of a species would seem to be impossible since the population of any species always changes by integer amounts. Hence, the population of any species can never be a differentiable function of time. If a given population is very large and it is suddenly increased by one unit, then the difference is very small when compared to the given population. Therefore, we make the approximation such that the large populations change continuously and even differentiable with time. The projection of future populations is normally based on present population and reliable growth rate and the first order differential equations govern the growth of various species. Further, in any species un controlled exponential growth is not viable as the population and the growth rate are limited by the various factors which determine the sustainability and the growth rate. Hence, in this paper, mathematical modelling for global population growth based on exponential and logistic growth models is presented. The use of these growth models is widely established in many fields of modelling and forecasting.

II. MATERIALS METHOD

A research is best understood as a process of arriving at dependent solutions to the problems through the systematic collection, analysis and interpretation of data. In this paper, secondary population of the world from 1960 to 2010 (inclusive) were collected from UN World Population Prospectus (2017 Revision). The exponential and logistic growth mathematical model was used to compute the predicted population values.

III. DEVELOPMENT OF THE METHOD

The Exponential Growth Model:

In 1798 Thomas R. Malthus proposed a mathematical model of population growth. He proposed by the assumption that the population grows at a rate proportional to the size of the population. This is a reasonable assumption for a population of a bacteria or animal under ideal conditions (unlimited environment, adequate nutrition, absence of predators, and immunity from disease).

Supposing we know the population P_0 at some given time $t=t_0$, and we are interested in protecting the population P, at some future time $t=t_1$, i.e. to find the population function P(t) for $t_0 \leq t \leq t1$ that satisfies $P(t_0)=P_0$.

Then considering the initial value problem

$$\frac{dP}{dt} = kP(t)\ t_0 \leq t \leq t_1; P(t_0) = P_0 \quad (1)$$

Integrating by variable separable in (1)

$$\int \frac{dP}{P} = k\int dt$$

$$\ln P = kt + c$$

$$P(t) = P_0 e^{kt} \text{ or}$$

$$P(t) = P_0 \exp\{k(t - t_0)\} \quad (2)$$

Where k is a constant known as the Malthus factor, is the multiple that determines the growth rate. Equation (1) is the Exponential growth model with (2) as its solution. It is a differential equation because it contains an unknown function P and it derivative . Having formulated the model, we now look at its consequences. If we rule out a population of 0, then P(t)>0 for all t. So, if k>0 then equation shows that for all t. This means that the population is always increasing. In fact, as P(t) increases, equation (1) shows that becomes larger. In other words, the growth rate increases as the population increases. Equation (1) is appropriate for modelling population growth under ideal conditions, thus we have to recognize that a more realistic must reflect the fact a given environment has a limited resource.

The Logistic Growth Model:

A Belgian Mathematician Verhulst, showed that the population growth not only depends on the population size but also on how far this size is from its upper limit i.e. its carrying capacity (maximum supportable population). He modified Malthus's Model to make the population size proportional to both the previous population and a new term

$$\frac{a - bP(t)}{a} \quad (3)$$

Whereaand b are called the vital coefficients of the population. This term reflects how far the population is from its maximum limit. However, as the population value grows and gets closer to $\frac{a}{b}$, this new term will becomesmaller and get to zero, providing the right feedback to limit the population growth. Thus, the second term models the competition for available resources, which tends to limit the population growth. So, the modified equation using this new term is:

$$\frac{d}{dt}P(t) = \frac{aP(t)(a - bP(t)}{a} \quad (4)$$

This is a nonlinear differential equation unlike equation (1) in the sense that one cannot simply multiply the previous population by a factor. In this case the population P(t) on the right of equation (4) is being multiplied by itself. This equation is known as the logistic law of population growth. Putting P = P0 for t = 0, whereP0 represents the population at some specified time, t = 0, equation (4) becomes

$$\frac{d}{dt}P = aP - bP \quad (5)$$

Separating the variables in equation (5) and integrating, we obtain

$$\int \frac{1}{a}\left(\frac{1}{P} + \frac{b}{a - bP}\right)dP = t + c \text{ so that}$$

$$\frac{1}{a}(\log P - \log(a - bP)) = t + c \quad (6)$$

Using t=0 and P=P0, we see that

$$c = \frac{1}{a}(\log P0 - \log(a - bP0))$$

$$\frac{1}{a}(\log P - \log(a - bP)) = t + \frac{1}{a}\log P0 - \log(a - bP0)$$

Equation (6) becomes, Solving for P gives

$$P = \frac{\frac{a}{b}}{1 + (\frac{\frac{a}{b}}{N_0} - 1)e^{-at}} \qquad (7)$$

Taking the limit of equation (7) as t→∞, we get (science a >0)

$$P_{\max} = \lim_{t\to\infty} P = \frac{a}{b} \quad (8)$$

If the time t=1 and t=2, then the values of P are P1 and P2 respectively, from (8) we get,

$$\frac{b}{a}(1 - e^{-a}) = \frac{1}{P1} - \frac{e^{-a}}{P_0}, \frac{b}{a}(1 - e^{-2a}) = \frac{1}{P_2} - \frac{e^{-2a}}{P_0} \quad (9)$$

Dividing the members of the second equation in relation (9) by corresponding Members of the first equation to eliminate $\frac{b}{a}$ we get

$$1 + e^{-1} = \frac{\frac{1}{P_2} - \frac{e^{-2a}}{P_0}}{\frac{1}{P_1} - \frac{e^{-a}}{P_0}} \quad (10)$$

So that,

$$e^{-a} = \frac{P_0(P_2 - P_1)}{P_2(P_1 - P_0)} \quad (11)$$

Substituting the values of e^{-a} in to the (9) equation, we obtain

$$\frac{b}{a} = \frac{P_1^2 - P_0P_2}{P_1(P_0P_1 - 2P_0P_2 + P_1P_2)} \quad (12)$$

Thus, limiting the value of P,we get

$$P_{\max} = \lim_{t\to\infty} P = \frac{a}{b} = \frac{P_1(P_0P_1 - 2P_0P_2 + P_1P_2)}{P_1^2 - P_0P_2} \quad (13)$$

IV. MAPE

The absolute percentage errors shown on the following worksheet are rounded to the nearest whole percentage. The MAPE for each forecast is calculated as the average of the percentages in the row based on the true percentages, as opposed to the rounded percentage. As a result, there may be a slight difference between the MAPE shown on the worksheet and the average of the rounded percentages.

Exponential Model	logistic model	Actual	Mape-Exponential model	Mape-logistic model
3033213	3033213	3033213	0	0
3627517.167	3700578	3700578	1.974308682	0
4338264.671	4458412	4458412	2.694845802	0
5188270.514	5296356.151	5330943	2.676308595	0.648794201
6204819.892	6196021.903	5751474	7.882255783	7.729286498
7420544.049	7131957.891	6145007	20.75729205	16.06102143
8874467.744	8074208.757			
10613261.94	8992008.315			
12692741.95	9857685.994			
15179659.1	10649773.69			
18153843.46	11354637.01			
21710766.38	11966516.13			
25964605.1	12486345.86			
31051907.91	12919943.96			
37135977.27	13276113.14			
	Mean absolute error %		5.997501819	4.073183688

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V. RESULTS

For the estimation of the future population of the world, we need to determine the growth rate of the world using exponential growth model in (2). Using the actual population of the world in thousands with t=0, we have $P_{0=}$3033213, P_1=3700578 when t=1. We can solve for the growth rate k

$$3700578 = 3033213e$$

$$k = \ln(\frac{3700578}{3033213})$$

$$k = 0.178925985$$

$$P(t) = 3033213e$$

Hence the general solution,
0.178925985k
This means that the predicted growth rate of the world population is 17.8 % for ten years. For the exponential growth model in (2) with this we projected the population of the world to 2050.
Based on table 1, let t=0,1and 2 correspond to the year 1960, 1961 and 1962 respectively.
Then and correspond to 3033213,3700578 and 4458412.
Substituting the values of P0, P1, and P2 into equation (13) we get

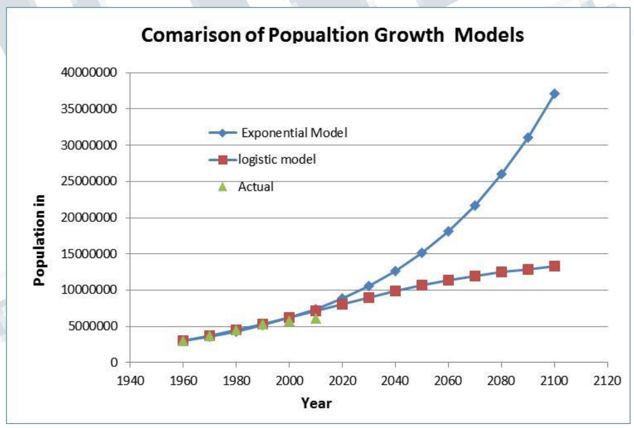
$$P_{\max} = \frac{a}{b} = 14647743$$

So therefore a=-ln (0.772562)
Solving this for a, we get
a = 0,258043.
This implies that the predicted growth rate of the world's population is 25.8% for ten years.
From a/b =14647743
We have b=1.7*10^8
Substituting the values of P_0 e^-a,a/b in (7) gives

$$P = \frac{14647763}{1 + (3.829118)(0.772562)^t}$$

Years	Actual Population	Projected Population	
		Exponential Growth	Logistic Growth
1960	3033213	3033213	3033213
1970	3700578	3627517.167	3700578
1980	4458412	4338264.671	4458412
1990	5330943	5188270.514	5296356.151
2000	5751474	6204819.892	6196021.903
2010		7420544.049	7131957.891
2020		8874467.744	8074208.757
2030		10613261.94	8992008.315
2040		12692741.95	9857685.994
2050		15179659.1	10649773.69
2060		18153843.46	11354637.01
2070		21710766.38	11966516.13
2080		25964605.1	12486345.86
2090		31051907.91	12919943.96
2100		37135977.27	13276113.14

Thus, from the above information it clearly states that logistic growth model is preferable to exponential growth model.



VI. CONCLUSION

In conclusion we found that the predicted carrying capacity for the population of the world is 37135977.27. Vital coefficient also plays an important role in population growth. Thus, the population growth rate of the world, according to this model, is % per annum. This approximated population growth rate compares well with the statistically the world's population growth in 2100. Based on this model we also found out that the population of the world is expected to be 13276113.14 in the year 2100.

The following are some recommendations:
Technological developments, pollution and social trends have significant influence on the vital coefficients a and b.

Therefore, they must be re-evaluated every few years to enhance the determination of variations in the population growth rate. In order to derive more benefits from models of population growth, one should subdivide populations into different age groups for effective capture, analyses and planning purposes. The government should work towards industrialization of the country for the attainment of vision 2020.

This will have an effect in improving its absorptive capacity for development through population growth rate measures. The more industrialized a Nation is, the more living space and food it has and the smaller the coefficient b, thus, raising the carrying capacity. However, present attempts appear to provide acceptable predictions for the world's population growth.

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