ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)

Vol 10, Issue 11, November 2023

Exploring and Understanding the Conic Section Curves and Its Beyond

[1] Tolga KABACA, [2] Kyeongsik CHOI

[1] Matematik Lezzeti Private Turoring Center, Turkey
[2] Mokwon University, Korea
Corresponding Author Email: [1] tolgakabaca@gmail.com, [2] kyeong@mokwon.ac.kr

Abstract— This study proposes a learning environment based on emergent modeling approach. The modeling task is "Creating a virtual and dynamic model of ellipse and hyperbola to reinvent the relationship between the parameters mentioned in the synthetic definitions of the curves and the parameters of the formal equation". It was explained that how the modeling task could help learners to reinvent some interesting relationships between ellipse and hyperbola. This paper also presented how students studied on ellipse and hyperbola in the proposed environment and reported their reflections.

Index Terms— Conic sections, Dynamic Mathematics, Emergent Modeling Ellipse, Hyperbola.

I. INTRODUCTION

Ellipse and Hyperbola's synthetic definitions are similar to each other. Two fixed points on plane and a parameter representing "total distance" or "distance difference" are mentioned in the definition. The locus of a point, whose total distance to two fixed point on plane is constant, is ellipse. Similarly, if the term "total distance" is changed into "distance difference" the statement defines hyperbola. Students generally learn these curves distinctly and based on their analytical equations. This paper's target is showing how emergent modeling approach may help emerging a formal mathematical knowledge of these two curves and how this approach help students to develop better understanding as well.

Actually, emergent modeling approach defines a modeling process, which starts from a real life situation, reaching a mathematical concept in formal at the end [1]. On the other hand, this approach is based on the idea of presenting an opportunity of reinvention environment for mathematical facts to students [2]. In this context, this study focused on reinvent mathematics and created a virtual examining environment instead of fully real life.

Article was designed under two steps. At the first step, a virtual modeling process, on GeoGebra the dynamic mathematics software was defined. In the process, ellipse and hyperbola were completely modeled virtually according to their oral definitions. Then, an algebraic examination was held in terms of virtual model's parameters. At last, it was explained that how virtual model emerged the formal knowledge about ellipse. By this way, we aimed to diversify the students' cognitive processes which are involved in geometrical reasoning as visualization processes, construction processes and reasoning processes, as it is highlighted and proposed by Duval [3].

At the second step, virtual modeling process was presented to a group of Korean high school students and their reactions, during studying on suggested learning environment, was reported.

II. CREATING THE VIRTUAL MODELS

A. Construction process of ellipse

Ellipse is a curve, which is defined as the set of the ordered pairs whose sum of the lengths to the two fixed points on the plane is constant.

Let the fixed points on the plane be F_1 and F_2 which are called focus points. It is advised to create a line connecting F_1 and F_2. Then, we need to create a point whose total length to F_1 and F2 is constant. So, we have to decide a constant value k representing the total length (or difference) to the focus points. Hereafter, following procedure may help to create the simulation;

- Create a line passing through two free points named F_1 and F_2. These points will represent the focuses.
- Create a circle centered at one of the focuses. It was preferred to create the circle passing through the free point G_1 on the line connecting the focus points. This circle's radius measured and named as r_1. The point G_1 will be called as generator point for ellipse.
- Create a slider k which has positive values.
- Create a second circle centered at other focus with the radius $k - r_1$.
- By this way, the sum of the lengths of any intersection point of two circles to the focus points will be the constant value k represented by the slider. The situations, which the circles don't coincide, means the position of focus points and the value of the k could not produce an ellipse. Just set the conditions as available for intersection of circles.
- Intersection points of the circle were named as E and E′. Then the traces of the E and E′ were activated to see the ellipse (see Figure 1). The ellipse will be observed when the generator point G_1 is shifted

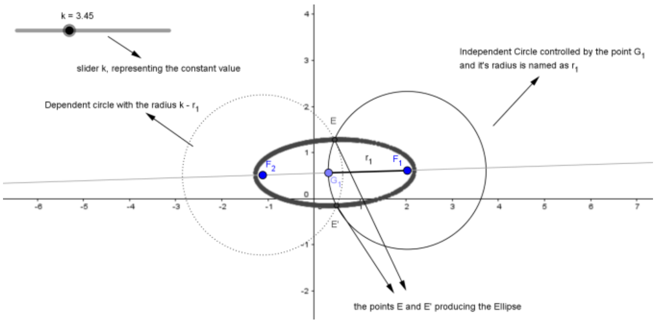

Figure 1: Virtual Model of Ellipse

Now, the focus points can be dragged the constant parameter k can be changed completely free. For some positions of the focus points and some values of the parameter k, it can be observed that the curve is not obtained. This situation can be explained as "distance between the focus points must not be greater than the value of the parameter k to obtain the ellipse".

B. Construction process of hyperbola

Lets' hide the circle centered at F_2 with the radius $k - r_1$. Then, create a new circle again centered at F_2 with the radius length $k + r_1$. By this way, the difference of the distance between the intersection point to F_2 and F_1 is constant k. So, the trace of the intersection points will create the hyperbola (see Figure 2).

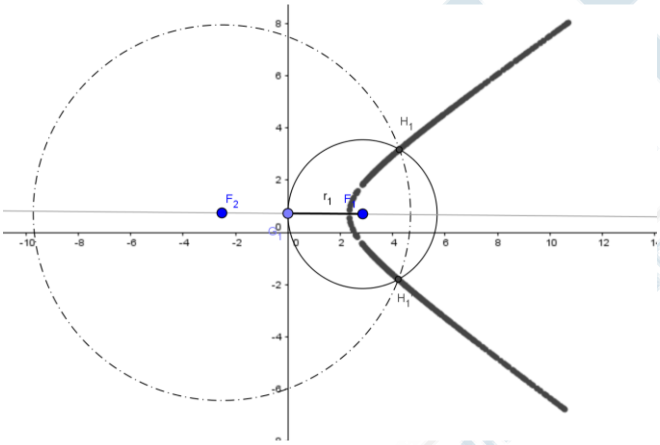

Figure 2: Virtual Model of Hyperbola (Part-1)

When the definition of hyperbola assessed again, we can conclude that the same procedure should be applied for other arm of the curve. That is, a free circle centered at F_2 with the radius r_2 and passing through the point G_2 should be created, then a dependent circle centered at F_1 with the radius $k + r_2$ should be obtained (see Figure 3) by dragging the generator point.

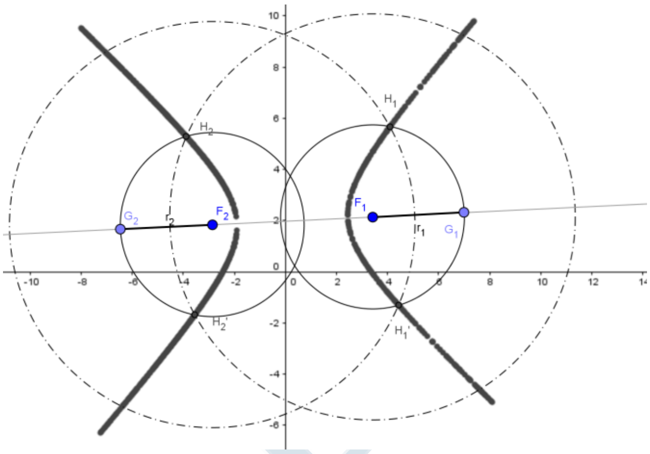

Figure 3: Virtual Model of Hyperbola (Part-2)

Again for some position of the focus points and some values of the parameter k, it is observed that the hyperbola curve cannot be obtained. This situation also can be explained as "distance between the focus points must not be smaller than the value of the parameter k to obtain the hyperbola".

As a result, we can understand that the conditions of obtaining two curves are complementary for each other. In fact, in the dynamic simulation; it is also observed that, when the ellipse cannot be obtained, the hyperbola can be obtained or vice versa.

When we consider on why this situation has been met, it can be understood that it is related with the relationship between the parameter k representing the constant sum or difference to the focus points and the length between the focus points. The curve will be ellipse, when the value of k greater than the length between the focus points. The curve will be hyperbola, when the value of k smaller than the length between the focus points.

III. FROM VIRTUAL MODELS TO ALGEBRAIC INFERENCES

At this step, it is wondered that the algebraic representation of our completely geometrically simulated model in terms of the focus points' analytic components and the parameter k. This was an excited exploration process, because even if a constant is mentioned in the definition of ellipse and hyperbola, this constant could not be seen in the formal equations of these conics. Before proceeding, the focus points were redefined in terms of GeoGebra sliders as free parameter. Furthermore, it is decided to place the focus points on x-axis symmetrically. By this way, the algebraic exploration has been made easier. Table-1 summarizes the algebraic exploration by defining two focus points as $(f_1, 0)$ and $(f_2, 0)$.

We obtained exactly same equation for both curves. This situation also helps us to understand why two curves were found as complementary for each other in the simulation model. When the equation-1 is simplified and re-organized, following form will be obtained.

$$\frac{-4x^2}{(f_1+f_2)^2-k^2}+\frac{4k^2y^2}{((f_1+f_2)^2-k^2)((f_1-f_2)^2-k^2)}+\frac{4(f_1+f_2)x}{(f_1+f_2)^2-k^2}=1$$

After entering equation-1 or equation-2 into GeoGebra input bar, it can be seen that traces obtained by virtual model and the curves obtained by the equations coincide both for ellipse and hyperbola. Note that, the focus points should be redefined as $(f_1, 0)$ and $(f_2, 0)$ and two sliders f_1 and f_2 must also be created before writing the equation above on GeoGebra.

Before proceeding, let's transform the equations into more well-known form, which the focuses are symmetric as $(f_1, 0)$ and $(-f_1, 0)$. When the second focus $(f_2, 0)$ is assigned as $(-f_1, 0)$, the term $f_1 + f_2$ will be 0. Then, the equation-2 will be transformed into shorter form as following for a particular form of ellipse and hyperbola;

$$\frac{x^2}{k^2/4}+\frac{y^2}{(k^2-4f_1^2)/4}=1$$

As it is well-known, the coordinate axes cut points are clearly visible in the formal equations. Hence, exploring the coordinate axes cut points of our ellipse equations, which is written in terms of virtual model parameters, was a remarkable effort. Any point, on the curve, has the distance of r_2 and r_1 to the focus points $F_2(-f_1, 0)$ and $F_1(f_1, 0)$ correspondingly. Therefore, $r_1+r_2 = k$ where k is the parameter representing the total length of any point to the focuses. Let's call the y-intercept of the curve as b and x-intercept of the curve as a (Figure-4). Additionally, remember that "the curve will be ellipse, when the value of k greater than the length between the focus points" that is $k > 2f_1 \Rightarrow k_2 > 4f_1^2$ in terms of virtual model parameters.

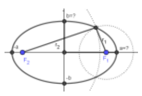

Figure 4: Standard Ellipse centered at origin

The point (0, *b*) has the same length, which is $r_1=r_2$ to the focus points F_1 and F_2 which is represented as $(f_1, 0)$ and $(-f_1, 0)$ correspondingly. Since $r_1+r_2=k$ and $r_1=r_2$, $r_1=k/2$. So, by using Pythagoras theorem, it can be reached that

$$b=\sqrt{\frac{k^2}{4}-f_1^2}=\sqrt{\frac{k^2-4f_1^2}{4}}.$$

The length of the point (*a*, 0) and the focus F_2 is r_2 which can be written as $2f_1+r_1$ at the same time. So, the equation $2f_1+r_1 = r_2 \Rightarrow r_1 - r_2 = 2f_1$ is can be written as well. When the system of equation $\{r_2 - r_1 = 2f_1, r_1+r_2=k\}$ has been solved, it was reached that

$$r_2=\frac{2f_1+k}{2}.$$

Since $a = r_2 - f_1$, it is reached that

$$a=\frac{2f_1+k}{2}-f_1=\frac{k}{2}.$$

So, it can be easily seen that the ellipse, which is centered at origin, can be represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where a and b are represent the coordinate axes cut points as (*a*, 0), (-*a*, 0), (0, *b*) and (0, -*b*).

IV. STUDYING WITH STUDENTS

After discovering the interesting relationships between ellipse and hyperbola by dynamic models, we designed a worksheet for our high school students. We encouraged the students to study with help of the worksheet and we reported their thoughts and reactions after cross-checking their written responses as two mathematics educators, one of us was students' mathematics teacher.

A. Participants

The participants of this study were 90, who were 10th-grade students of A high school in Korea. Students did not know how to use GeoGebra and had limited, mostly procedural knowledge, experience of learning about quadratic curves, such as parabolic, elliptic, or hyperbolic curves. These students were in the school for the gifted with advanced learning skills in Korea. By this choice of participants, we aimed to see the effect of dynamic modeling purely, rather than students' own learning difficulties. Whole learning activity was conducted during three classes, each class consisting of two hours of block time. Students participated in activities using GeoGebra to write their worksheets.

B. Lesson Descriptions

The first lesson consists of time to learn the general usage of GeoGebra. Most of the students in this study had no experience using GeoGebra. Therefore, in the first lesson, students were explained the geometric and algebraic functions of GeoGebra, and the students were able to adapt to GeoGebra to some extent through simple exercises.

In the second lesson, modeling activities were performed in GeoGebra for the parabola curve, one of the quadratic curves. The geometric definition of the parabola curve is provided, and the algebraic relationship is derived from it. We assumed this session as pilot session to make students more familiar with the use of GeoGebra. So, this session is not part of current article.

In the third lesson, modeling activities were performed in GeoGebra for ellipses and hyperbolas among the quadratic curves. Students studied under the guidance of worksheet (appendix-1) which is designed to help students on determining appropriate variables and living the similar experience as it is defined in the previous section. Furthermore, this worksheet asked students describing the commonalities and differences between ellipses and hyperbolas and summarizing what they learned and felt from this activity.

C. Reactions of the Students

Only about one-third of all students found that despite guidance in a given worksheet, ellipse and hyperbola shared the same expression. This is thought to be because, in Korea, students ignored the order in which they were guided to the worksheet by learning about ellipse and hyperbola lines in advance through private education. However, the students in the order of the worksheet found the correct expression and said they were surprised that the equation of the ellipse and the hyperbola can be expressed in the same way. We found some evidences that students, who can find out the expected relations, were positively affected from the exploration process under dynamic modeling.

After realizing that equations of ellipse and hyperbola were getting similar, Student S1 wrote that "*At last, it was interesting that the two different symbols become squared and eventually the same.*" S1 also reflected his/her opinions by declaring that "*Drawing ellipse and hyperbola was the most memorable experience. The best in the activities were inducing the equations and thinking about the relationship between them. By directly squaring the formula, there was a chance to think more about what the principle was in it, and to learn interesting things, such as that it eventually came from the same equation.*"

Similarly, another student S2, reflected his/her thought about how he/she was affected by finding out the connections between ellipse and hyperbola by writing "So far, *I have drawn a lot of ellipse and hyperbola and memorized formulas, but I have never thought about any relevance. But through this class, I had a chance to think about the connection. Later, I will look for more relevance between the two curves in the process of direct formulation*". Student S3 declared that he/she was affected like experiencing a magic by saying "*I was surprised that the expression of the ellipse and hyperbola was the same as a result. It was like seeing real magic in front of my eyes. I was able to tell if it was hyperbolic or elliptical depending on the value (of a variable). So I thought I could use GeoGebra to explore equations and access new equations. I would like to graph another equation using GeoGebra.*"

Student S4 and S5 mostly focused on the advantages of dynamic activity. They expressed their feeling as "*I find it helpful to see the graphs in GeoGebra and draw equations directly. I also found that ellipses and hyperbolas have the same equation. Also, I can find the range of x by changing the focal range x coordinate that can identify the ellipse and hyperbola*" and "*I liked using GeoGebra to find the relationship between ellipse and hyperbola. When I entered the formula in GeoGebra, I was able to see the graph. And I saw how the graph changes as the value of slider k keeps changing. I liked this exploration activity*" respectively.

Student S6 reflected his/her affective view as following "*I derived both equations from the definition of ellipse and hyperbola and found that both equations are the same. So I was wondering why they are the same. I squared both sides in the process of deriving the equation and found that the equation had the same shape. When I entered the equation in GeoGebra, the graphs appeared differently as the values of the variables changed. In drawing this graph, I was amazed at the beauty of mathematics.*"

Student S7 and S8 declared the importance of visual understanding instead of memorizing and a valuable learning experience for the future: "*I previously memorized the graph representation of the ellipse and hyperbola. However, this activity gave me a visual understanding of the principles of constructing ellipses and hyperbolas in GeoGebra. In the future, I will try to understand other graphs using traces of the curve*" and "*I was surprised that I got the same expression when formulating using the definition of ellipse and hyperbola. I have often read textbooks and reference books, but I have never thought about this relationship. Also, it was good to understand the ellipse and hyperbola as traces of points through this activity*" and "*When I derived the elliptic formula and then entered the equation in GeoGebra, I was puzzled because a hyperbola rather than an ellipse was drawn. However, after adjusting the variable k value, it was found that the hyperbola is transformed into an ellipse. Through this activity, I learned that there is a visual connection between the hyperbola and the ellipse. I think this activity is very important for understanding the relationship between the two curves*" respectively.

V. CONCLUSION

This article defined an emergent modeling task by starting from virtual environment instead of real situation. Since the proposed modeling task satisfied the condition of reinventing mathematics, emergent modeling definition has been seen appropriate.

The task was started from conceptual definition of ellipse and hyperbola. Any other formal knowledge has been assumed that unknown along the complete task. First, the curves were obtained by using fundamental geometry knowledge in dynamic Mathematics software environment. The product of first step was just a locus of a particular point (trace observation property of GeoGebra). This curve was just a heuristic result and it is observed that a particular relationship among distance, between focus points and the parameter k, defining the total distance to focus points, determined the conic type. Second, the analytical equations was tried to be wrote in terms of the parameters used in the software. The emerged equation was unique for two conic curves. So, it was explored that a particular relationship among the parameters will determine the equation defines which curve. This relationship was just algebraic version of the relationship explored in the first step.

IFERP
connecting engineers... developing research

ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)

Vol 10, Issue 11, November 2023

Before the third step, it was decided to restrict the equation into a particular form of ellipse and hyperbola which is centered at origin, in other words “the focus points are symmetric according to origin and located on x-axis”. After this restriction, following equation was obtained

$$\frac{x^2}{k^2/4}+\frac{y^2}{\left(k^2-4f^2\right)/4}=1$$

In this equation, k was representing the total distance of the locus points to the focus points for ellipse and it was representing the distance difference of the locus points to the focus points for hyperbola. The parameter f was representing the focus points (f, 0) and (-f, 0). According to this parameterization, if k > 2f then the curve was an ellipse, if k < 2f then the curve was a hyperbola.

At the third step, a geometric examination was defined. The equation was tried to be revised in terms of the points which the curve cut the coordinate axes. This examination was also restricted to ellipse only. The product of this examination was the relationship between the points, which ellipse cut the coordinates axes and the parameters k and f representing the total distance of the locus point to the focus points and focus points respectively. This result was

$$a=\frac{k}{2}\ ,\quad b=\sqrt{\frac{k^2-4f^2}{4}}$$

where a and b are the positive cut point on x-axis and the positive cut point on y-axis respectively. So, following formally known equation of standard ellipse, centered at origin has been also reached.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The defined environment has been conducted to Korean high school students who do not know these conic curves before except their partial knowledge from private courses. Even if meaningful number of students could not show desired understanding indication, maybe sourced by past learning routines, it is observed that this learning environment have the potential of “emotionally affecting the students”, “making the students to develop new attitudes for future learning” and “appreciating the dynamic software activity”. We also reached the conclusion that dynamic visualization has the potential of enforcing the understanding the similar concepts as suggested by previous research findings [4], [5].

Table-1: Summary of the algebraic exploration by defining two focus points as (f_1, 0) and (f_2, 0)

Ellipse	Hyperbola	Explanation
$\sqrt{(x-f_1)^2+y^2}+\sqrt{(x-f_2)^2+y^2}=k$	$\left\lvert\sqrt{(x-f_1)^2+y^2}-\sqrt{(x-f_2)^2+y^2}\right\rvert=k$	Equations coming from the definitions
$(x-f_1)^2+y^2+(x-f_2)^2+y^2+2\sqrt{\left((x-f_1)^2+y^2\right)\left((x-f_2)^2+y^2\right)}=k^2$	$(x-f_1)^2+y^2+(x-f_2)^2+y^2-2\sqrt{\left((x-f_1)^2+y^2\right)\left((x-f_2)^2+y^2\right)}=k^2$	Take the square of both sides of the equations
$2\sqrt{\left((x-f_1)^2+y^2\right)\left((x-f_2)^2+y^2\right)}=k^2-(x-f_1)^2-(x-f_2)^2-2y^2$	$-2\sqrt{\left((x-f_1)^2+y^2\right)\left((x-f_2)^2+y^2\right)}=k^2-(x-f_1)^2-(x-f_2)^2-2y^2$	Place the square rooted term in the left hand side of the equation
$4\left((x-f_1)^2+y^2\right)\left((x-f_2)^2+y^2\right)=\left(k^2-(x-f_1)^2-(x-f_2)^2-2y^2\right)^2$	$4\left((x-f_1)^2+y^2\right)\left((x-f_2)^2+y^2\right)=\left(k^2-(x-f_1)^2-(x-f_2)^2-2y^2\right)^2$	Take the square of both sides again *(Equation-1)*

REFERENCES

[1] Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1(2), 155–177

[2] Freudenthal, H. (1991). Revisiting mathematics education—China lectures. Dordrecht: *Kluwer Academic Publishers*

[3] Duval R (1998). Geometry from a Cognitive Point of View. In C Mammana and V Villani (Eds), *Perspective on the Teaching of Geometry for the 21st Century: an ICMI study*. Dordrecht: Kluwer.

[4] Edwards JA, Jones K (2006). Linking Geometry and Algebra with Geogebra, *Math. Teach.* 194:28-30.

[5] Gorgorio N, Jones K (1996). Elements of the Visualisation Process within a Dynamic Geometry Environment. *8th International Congress of Mathematical Education*, Spain. p. 6.