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# Algorithm for Bessel Function Evaluation of a Complex Number

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***Abstract***— ***This paper outlines an original algorithm for Bessel functions of complex values. Bessel functions of the first kind, with an integer index, $J_n(x)$, when using the standard math or complex headers in C++, are only defined for real numbers. My algorithm uses both the power series representation of $J_n(x)$, for x near the origin, otherwise, a Taylor series of $J_n(x)$. For the Taylor series of $J_n(x)$ about $x_0$, the kth derivative, $J_n^{(k)}(x_0)$, is written as $\left(a_0 + \sum_{m=1}^{k} a_m\, x_0^{-m}\right)J_{n-1}(x_0) + \left(b_0 + \sum_{m=1}^{k} b_m\, x_0^{-m}\right)J_n(x_0)$, where the coefficients can be calculated from the coefficients from lower derivatives.***

***Index Terms***— ***algorithm, Bessel function, complex valued, Taylor series***

## I. INTRODUCTION

The Bessel function has the properties [1]:

$$J_n(-z) = (-1)^n J_n(z)$$
$$J_n(\bar{z}) = \overline{J_n(z))}$$

Thus, one only needs to concentrate calculations for complex numbers with non-negative real and complex parts.

The power series of Bessel function [1], with an integer index, is

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{2k} \frac{(-1)^k}{k!\,(n+k)!}$$

Note that the above power series, converges for all complex numbers.

The following is the absolute value of the ratio of consecutive terms for the above power series.

$$\frac{\left|\dfrac{(-1)^{k+1}\left(\frac{z}{2}\right)^{2(k+1)}}{(k+1)!\,((k+1)+n)!}\right|}{\left|\dfrac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{k!\,(k+n)!}\right|} = \frac{|z|^2}{4(k+1)(k+1+n)}$$

Thus, when $|z| \le 2\sqrt{n+1}$, the Bessel power series has terms whose magnitude is always decreasing.

So, for all $n$, the power series will rapidly converge when $|z| \le 2\sqrt{n+1}$.

Note that this region of values is small; in fact, does not include even the first non-zero zero of $J_n(z)$.

## II. TAYLOR SERIES

Since $J_n(z)$ is a smooth function, we have its Taylor's series, expanded around a number $z_0$

$$J_n(z) = \sum_{k=0}^{\infty} \frac{J_n^{(k)}(z_0)(z-z_0)^k}{k!}$$

Using the identities

$$J_n'(z) = J_{n-1}(z) - \frac{n}{z} J_n(z)$$
$$J_{n-1}'(z) = -J_n(z) + \frac{n-1}{z} J_{n-1}(z)$$

we, inductively, see that we may write

$$J_n^{(k)}(z) = \left(a_0 + \sum_{m=1}^{k} a_m\, z^{-m}\right) J_{n-1}(z) + \left(b_0 + \sum_{m=1}^{k} b_m\, z^{-m}\right) J_n(z),$$

where $a_0, \dots, a_k$ and $b_0, \dots b_k$ are integers.

Notice that

$$\frac{d}{dz}\left(\left(a_0 + \sum_{k=1}^{q} a_k\, z^{-k}\right) J_{n-1}(z) + \left(b_0 + \sum_{k=1}^{q} b_k\, z^{-k}\right) J_n(z)\right) = b_0 J_{n-1}(z) - a_0 J_n(z) + J_{n-1}(z) \sum_{k=1}^{q} \left((n-k)a_{k-1} + b_k\right) z^{-k} + (n-q-1)a_q z^{-q-1} - J_n(z) \sum_{k=1}^{q} \left(a_k + (k+n-1)b_{k-1}\right) z^{-k} + (q+n)b_q z^{-q-1}$$

Thus, we can recursively write the $k^{th}$ derivative of $J_n(z)$ as

$$\left(a_{k,0} + \sum_{m=1}^{k} a_{k,m}\, z^{-m}\right) J_{n-1}(z) +$$

$$\left(b_{k,0} + \sum_{m=1}^{k} b_{k,m}\, z^{-m}\right) J_n(z)$$

Where $a_{0,0} = 0$ and $b_{0,0} = 1$,

$$a_{k,0} = b_{k-1,0} \text{ and } b_{k,0} = -a_{k-1,0},$$

$$a_{k,k} = (n-k)a_{k-1,k-1} \text{ and } b_{k,k} = -(k+n-1)b_{k-1,k-1}$$

And for $1 \le m < k$,

$$a_{k,m} = (n-m)a_{k-1,m-1} + b_{k-1,m}$$

$$b_{k,m} = -a_{k-1,m} - (m+n-1)b_{k-1,m-1}$$

In the same manner, we can recursively write the $k^{th}$ derivative of $J_{n-1}(z)$ as

$$\left(c_{k,0} + \textstyle\sum_{m=1}^{k} c_{k,m}\, z^{-m}\right) J_{n-1}(z) + \left(d_{k,0} + \sum_{m=1}^{k} d_{k,m}\, z^{-m}\right) J_n(z)$$

where

$$c_{0,0} = 1 \text{ and } d_{0,0} = 0,$$

$$c_{k,0} = d_{m-1,0} \text{ and } d_{k,0} = -c_{k-1,0},$$

$$c_{k,k} = (n-k)c_{k-1,k-1} \text{ and } d_{k,k} = -(k+n-1)d_{k-1,k-1}$$

And for $1 \le k < m$,

$$c_{k,m} = (n-m)c_{k-1,m-1} + d_{k-1,m}$$

$$d_{k,m} = -c_{k-1,m} - (m+n-1)d_{k-1,m-1}$$

Therefore, we have the new Bessel series

$$J_n(z) = \sum_{k=0}^{\infty} \frac{\left(\left(a_{k,0} + \sum_{m=1}^{k} a_{k,m}\, z_0^{-m}\right) J_{n-1}(z_0)\right)(z-z_0)^k}{k!} + \sum_{k=0}^{\infty} \frac{\left(\left(b_{k,0} + \sum_{m=1}^{k} b_{k,m}\, z_0^{-m}\right) J_n(z_0)\right)(z-z_0)^k}{k!}$$

$$J_{n-1}(z) = \sum_{k=0}^{\infty} \frac{\left(\left(a_{k,0} + \sum_{m=1}^{k} a_{k,m}\, z_0^{-m}\right) J_{n-1}(z_0)\right)(z-z_0)^k}{k!} + \sum_{k=0}^{\infty} \frac{\left(\left(b_{k,0} + \sum_{m=1}^{k} b_{k,m}\, z_0^{-m}\right) J_n(z_0)\right)(z-z_0)^k}{k!}$$

## III. ALGORITHM

**Bessel Evaluation Algorithm for $z$ in the First Quadrant in the Complex Plane**

**Algorithm 1**

If $|z| \le 2\sqrt{n+1}$
Use the Bessel power series.
Else
Set $z_0 = 2\sqrt{n+1}$
Use the Bessel power series to calculate $J_{n-1}(z_0)$ and $J_n(z_0)$.
While
$\mathrm{Re}(z) - z_0 > 0.5$
Set $z_1 = z_0 + 0.5$
Use $J_{n-1}(z_0)$ and $J_n(z_0)$, in the new Bessel series, to approximate $J_{n-1}(z_1)$ and $J_n(z_1)$.
Set $z_0 = z_1$.
Use $J_{n-1}(z_0)$ and $J_n(z_0)$, in the new Bessel series, to approximate $J_n(Re(z))$.
Set $z_0 = Re(z)$ and use the Bessel power series to calculate $J_{n-1}(z_0)$ and $J_n(z_0)$.
While $\mathrm{Im}(z) - Im(z_0) > 0.5$
Set $z_1 = z_0 + 0.5i$
Use $J_{n-1}(z_0)$ and $J_n(z_0)$, in the new Bessel series, to approximate $J_{n-1}(z_1)$ and $J_n(z_1)$. Set $z_0 = z_1$.
Use $J_{n-1}(z_0)$ and $J_n(z_0)$, in the new Bessel series, to approximate $J_n(z)$.

To speed up this algorithm, an array of values of $J_n(x)$, for $0 \le n \le 202$, where $x = 2(a+bi)$, $a$ and $b$ are integers such that $0 \le a \le 150$ and $0 \le b \le 150$, was calculated using multiple precision (65 digits).

**Algorithm 2**

Select integers $a$ and $b$ so that $2(a+bi)$ is closest to $z$.
Set $z_0 = 2(a+bi)$, $J_n(z_0)$ to be the element in row $b$ and column $a$ in the array for $J_n(z_0)$ and $J_{n-1}(z_0)$ to be the element in row $b$ and column $a$ in the array for $J_{n-1}(x)$.
Set $v_0 = (xz - z_0)/(2|z - z_0|)$.
While $|z - z_0| > 0.5$
Set $z_1 = z_0 + v_0$
Use $J_{n-1}(z_0)$ and $J_n(z_0)$, in the new Bessel series, to approximate $J_{n-1}(z_1)$ and $J_n(z_1)$.
Set $z_0 = z_1$.
Use $J_{n-1}(z_0)$ and $J_n(z_0)$, in the new Bessel series, to approximate $J_n(z)$.

## IV. STATISTICAL ANALYSIS

For an unbiased check for accuracy of the algorithm, the identity, $J_{n+2}(x) = 2(n+1)J_{n+1}(x)/x - J_n(x)$ was used to compare the algorithm against itself.

Two complex regions were investigated: $[0,300] \times [0i, 10i]$ and $[0,300] \times [10i, 300i]$. A random set of 1000 complex numbers were chosen from each region. For each random number, both sides of the equation were calculated and the two calculations were compared for the number of matching significant digits. The Bessel function's index was varied from 0 to 200. Calculations were done using python, in double precision and multiple precision with 34 digits.

The same set of random numbers were used for farther comparison, double precision of octave's Bessel function and

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python's (special from scipy), and multiple precision, 34 digits, of the calculator from the website keisan.casio.com and the mpmath python Bessel function.

## V. RESULTS

### Number of Agreed Digits

**Double Precision: Region 1**

| | Min. | Mean | Stand. Dev. |
|---|---|---|---|
| octave | 0 | 12.982 | 1.7695 |
| python | 0 | 13.120 | 1.8020 |
| algorithm | 0 | 14.119 | 1.2882 |

Note: All 0's occurred with high indexes and near the origin.

**Double Precision: Region 2**

| | Min. | Mean | Stand. Dev. |
|---|---|---|---|
| octave | 6 | 13.362 | 1.5039 |
| python | 7 | 13.383 | 1.6299 |
| algorithm | 8 | 14.434 | 0.84430 |

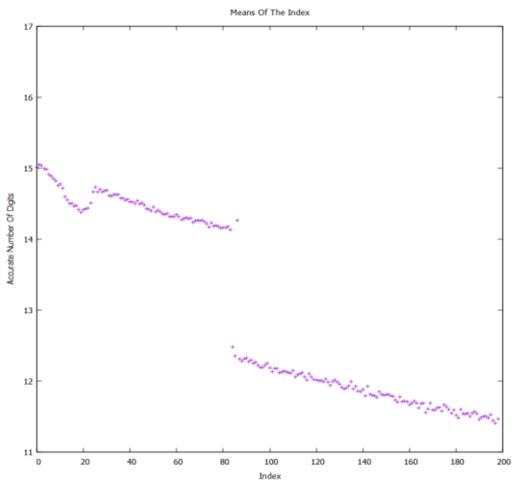
**34 Digits: Region 1**

| | Min. | Mean | Stand. Dev. |
|---|---|---|---|
| web | 10 | 30.325 | 5.32787 |
| python | 27 | 33.391 | 0.81832 |
| algorithm | 26 | 32.792 | 1.1198 |

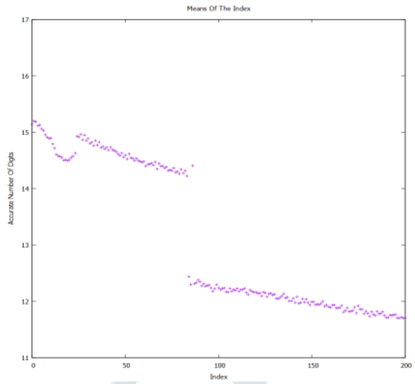
**34 Digits: Region 2**

| | Min. | Mean | Stand. Dev. |
|---|---|---|---|
| web | 11 | 31.391 | 4.6493 |
| python | 29 | 33.586 | 0.61193 |
| algorithm | 26 | 32.792 | 1.1198 |

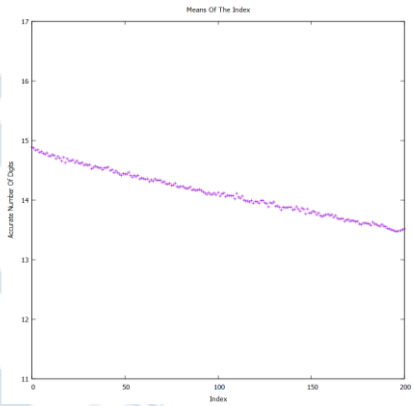
### Double Precision Accuracy: Region 1



Octave

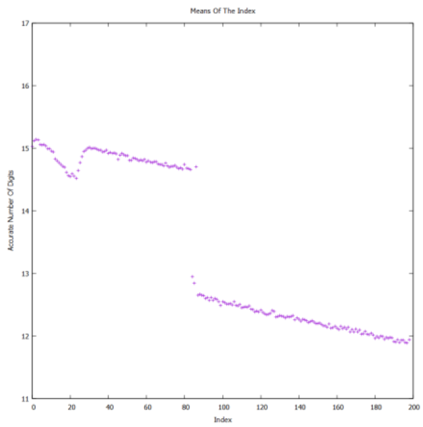


Python

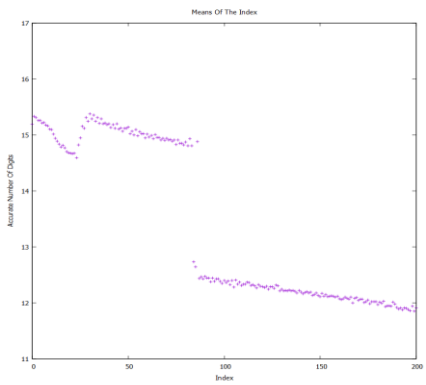


Algorithm

### Double Precision Accuracy: Region 2



Octave



Python

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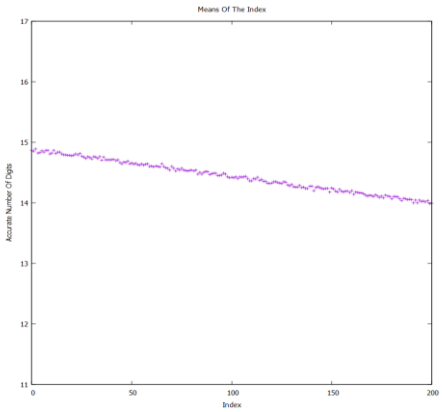
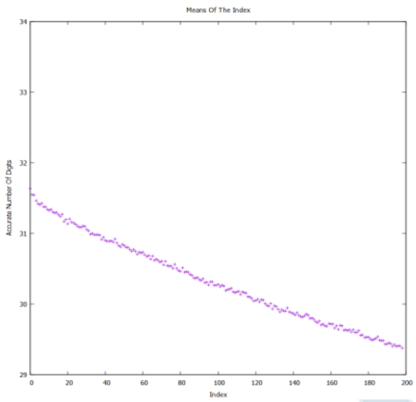
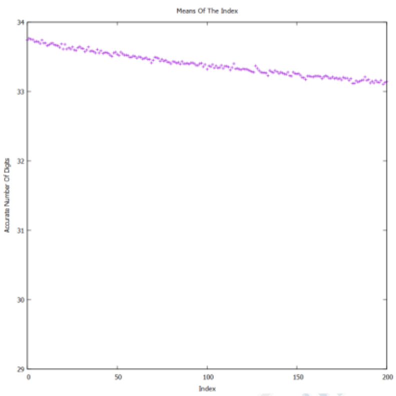



Algorithm

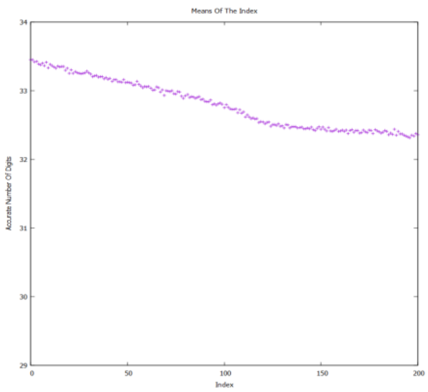
**34 Digit Accuracy: Region 1**




Web

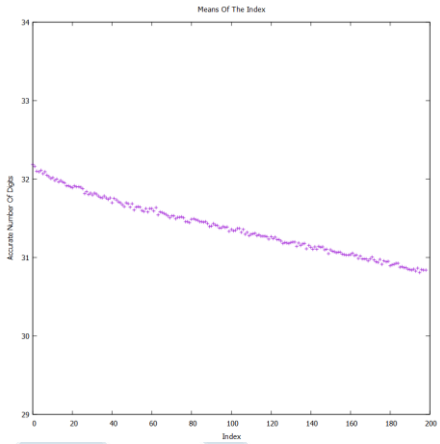



Python

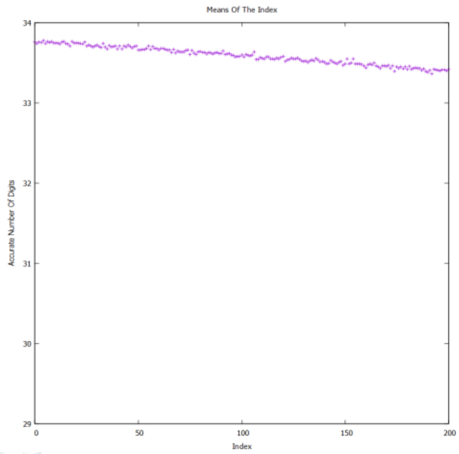



Algorithm

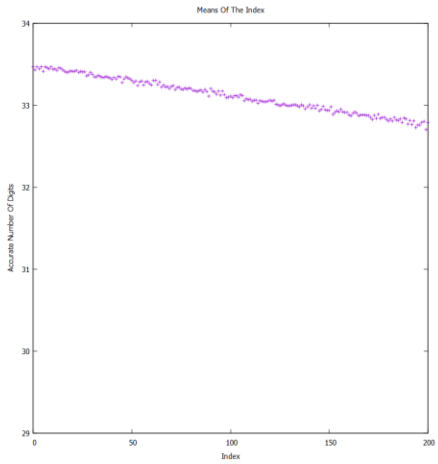
**34 Digit Accuracy: Region 2**




Web




Python




Algorithm

## VI. CONCLUSIONS

Concentrating on the effectiveness of each algorithm on preserving the identity $J_{n+2}(z) = \frac{2(n+1)}{z} J_{n+1}(z) - J_n(z)$, the analysis shows that, for double precision, my algorithm, compared to the Bessel algorithm for octave and python, is better and (95%) significantly better for the indexes greater than 88.

For 34-digit precision, the algorithm used by python was best, but not (95%) significantly better than either of the other algorithms.

All algorithms showed that increasing the index of the Bessel function had a negative influence on the accuracy.

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