ISSN (Online) 2394-2320

International Journal of Engineering Research in Computer Science and Engineering (IJERCSE)
Vol 5, Issue 2, February 2018

A New Approach to Fuzzy Bitopological Space via γ-Open Sets

[1] Birojit Das, [2] Baby Bhattacharya
[1] PhD Scholar,[2] Assistant Professor
[1][2] Department of Mathematics, NIT Agartala

Abstract **- B. C. Tripathy and S. Debnath [B. C. Tripathy, S. Debnath, γ-Open Sets and γ-Continuous Mappings in Fuzzy Bitopological Spaces, J. of Intelligence and Fuzzy Systems, 24, 631-635 (2013)] first introduced the concept of $(\mathrm{i,j})$ fuzzy γ-open set in a fuzzy bitopological space to show that the collection of all these sets forms a fuzzy topology and also studied fuzzy pairwise γ-continuity. In this present treatise, we redefine this set as fuzzy $(\mathrm{i,j})$ γ-open set with the help of fuzzy $(\mathrm{i,j})$ preopen set to show that it is completely independent from fuzzy $(\mathrm{i,j})$ open set. Moreover, we introduce fuzzy $(\mathrm{i,j})$ γ-generalized closed set as a generalization of fuzzy $(\mathrm{i,j})$ closed set and then we establish various properties and characterizations along with interrelationship among them. Also, we show an important result which states that though every fuzzy $(\mathrm{i,j})$ closed set is a fuzzy $(\mathrm{i,j})$ generalized closed set and fuzzy $(\mathrm{i,j})$ γ-generalized closed set but fuzzy $(\mathrm{i,j})$ generalized closed set and fuzzy $(\mathrm{i,j})$ γ-generalized closed set are totally independent of each other. Furthermore, we introduce the notion of $(\mathrm{i,j})$ fuzzy continuous function, $(\mathrm{i,j})$ gf continuous function, $(\mathrm{i,j})$ fuzzy γ-continuous function and $(\mathrm{i,j})$ γ-gf continuous function in a fuzzy bitopological space and study those functions with various properties and interrelationships. Lastly, we define a new type of closure operator and prove certain results based on this conception.**

Index Terms— Fuzzy (i,j) γ-open set, (i,j) gf closed set, (i,j) γ-gf closed set, (i,j) γ-gf continuous function.

I. INTRODUCTION

J. C. Kelly [9] in 1963 first introduced the concept of bitopological space. Kandil initiated the notion of fuzzy bitopological Space (FBTS) in the literature [1] and since then many concepts in bitopological space were extended to FBTS. In 1970, the notion of generalized closed set was first commenced by N. Levine [13] in topological space. T.Fukutake [18] (1986) extended this work in the field of bitopology. Later, in 1997 Balasubramanian and Sundaram [7] defined generalized fuzzy closed set in fuzzy topological space. On a different note N. Palaniappan and K. C. Rao [14] studied regular generalized closed sets in a topological space in 1993. Then, J. H. Park and J. K. Park [10] extended the results of Palaniappan et al. (regular generalized fuzzy closed sets) in fuzzy topological space. Considering the regular closure operator, S. Bhattacharya [16] introduced generalized regular closed sets in topological space in a different approach. That approach opened new windows for research in this sector. B. Bhattacharya and J. Chakraborty [3] continued this particular study of generalized regular fuzzy closed sets in fuzzy topological space. Moreover S. S. Kumar [17] has introduced the notion of (i,j) fuzzy preopen set in fuzzy bitopological space and considering this set, B. C. Tripathy and S. Debnath [6] studied the notion of (i,j) fuzzy γ-open sets and fuzzy γ-continuous mappings in fuzzy bitopological space. In this paper, we extend the notion of $(1,2)^*$ γ-open set [4] of bitopological space into FBTS and denote it as fuzzy (i,j) γ-open set. The work targets to study a generalized form of this fuzzy (i,j) γ-open set in FBTS and we call it (i,j) γ-generalized fuzzy closed set. At the same time, we introduce the notion of (i,j) generalized fuzzy closed set to study some interrelationships among these two ideas. Very often, it is observed that the relation between fuzzy closed set and generalized fuzzy closed set is linear in fuzzy topological space. For instance, every fuzzy closed set is generalized fuzzy closed [7] and every generalized fuzzy closed is a regular generalized fuzzy closed set but the converses are not true in general [10]. Here, it is shown that the relation between fuzzy (i,j) closed set, (i,j) generalized fuzzy closed set and (i,j) γ-generalized fuzzy closed set are of different kind and we emphasize that the last two concepts are independent of each other. It means that here the case is not normally abnormal, but abnormally normal as it is a natural offshoot of the context. Moreover, we present the notion of (i,j) generalized fuzzy continuous function and (i,j) γ-generalized fuzzy continuous function along with some properties. Eventually, we show that the notions of (i,j) generalized fuzzy continuity and (i,j) γ-generalized fuzzy continuity are independent of each other.

Throughout this paper, we denote a fuzzy bitopological space by FBTS which is given by (X, τ_i, τ_j), $i, j = 1, 2, i \neq j$ and is simply denoted as (X, τ_i, τ_j). Also fuzzy sets are here denoted by λ, μ, ν, η and fuzzy topologies are designated by $\tau_i, \tau_j, \mu_i, \mu_j, \rho_i, \rho_j$ for $i, j = 1, 2$ and $i \neq j$. Again, a fuzzy point in X with support $x \in X$ and value p $(0 \leq p \leq 1)$ is denoted by x_p, whose value is also represented by α , β. Also, for a fuzzy set λ in X, $1_X - \lambda$ denote the complement of the fuzzy set λ in X. Some important related definitions are recalled below as ready references of our research work.

1.1 Definition [1]: Let X be a non-empty set and τ_1, τ_2 be two fuzzy topologies defined on X. Then, (X, τ_1, τ_2) is said to be a FBTS.

1.2 Definition [5]: A fuzzy subset λ of a fuzzy topological space (X, τ) is said to be fuzzy γ^*-open if $\lambda \wedge \mu \in FPO(X)$ for each $\mu \in FPO(X)$, where $FPO(X)$ is the family of all fuzzy preopen sets in X.

1.3 Definition [12]: Let (X, τ_1, τ_2) be a bitopological space. Any subset A of X is said to be a $\tau_1\tau_2$-open set if $A \in \tau_1 \cup \tau_2$.

1.4 Definition [17]: A fuzzy subset λ of a FBTS is said to be (i, j) fuzzy preopen if $\lambda \leq i\text{-}int(j\text{-}cl(\lambda))$, where $i \neq j$ and $i, j = 1, 2$.

1.5 Definition [6]: A fuzzy subset λ of a FBTS (X, τ_i, τ_j) is called (i, j) fuzzy γ-open if $\lambda \wedge \mu$ is (i, j) fuzzy preopen for every (i, j) fuzzy preopen set μ in X. A fuzzy subset η of X is called (i, j) fuzzy γ-closed set if its complement, $1_X - \eta$ in X is a (i, j) fuzzy γ-open set.

1.6 Definition [7]: Let (X, τ) be a fuzzy topological space. A fuzzy set λ in X is called a generalized fuzzy closed set if $cl(\lambda) \leq \mu$, whenever, $\lambda \leq \mu$ and μ is a fuzzy open set in X.

1.7 Definition [2]: A fuzzy subset λ in a FBTS (X, τ_i, τ_j) is called a (i, j) fuzzy generalized closed set if $\tau_j\text{-}cl(\lambda) \leq \mu$, whenever $\lambda \leq \mu$ and $\mu \in \tau_i\text{-}FO(X)$, where $\tau_i\text{-}FO(X)$ is the family of all τ_i-fuzzy open sets.

1.8 Definition [7]: A map $f : X \to Y$ from fuzzy topological space (X, τ) into another fuzzy topological space (Y, σ) is called a generalized fuzzy continuous function if the inverse image of every fuzzy closed set in Y is generalized fuzzy closed in X.

II (i, j) FUZZY γ-GENERALIZED CLOSED SET

Here we consider the notion of $\tau_1\tau_2$-open set in a bitopological space [12] and extend this concept in a FBTS and call it fuzzy (i, j) γ-open set. With the help of fuzzy (i, j) open set, we define (i, j) generalized fuzzy closed set and (i, j) γ-generalized fuzzy closed set along with their interrelationships.

2.1 Definition: Let (X, τ_i, τ_j) be a FBTS and λ be any fuzzy subset of X. Then, λ is said to be fuzzy (i, j) open set if $\lambda \in \tau_i \vee \tau_j$. The complement of a fuzzy (i, j) open set is said to be a fuzzy (i, j) closed set. The collection of all fuzzy (i, j) open sets and fuzzy (i, j) closed sets are denoted by (i, j) $FO(X)$ and (i, j) $FC(X)$ respectively.

2.2 Definition: For a fuzzy set λ in X

(i) (i, j) closure of λ is the intersection of all fuzzy (i, j) closed sets containing λ and is denoted simply as (i, j) $cl(\lambda)$.

(ii) (i, j) interior of λ is the union of all fuzzy (i, j) open sets contained in λ and is denoted as (i, j) $int(\lambda)$.

2.3 Definition: A fuzzy subset λ in a FBTS (X, τ_i, τ_j) is said to be a fuzzy (i, j) preopen set if $\lambda \leq (i, j)\ int\big((i, j)\ cl(\lambda)\big)$. The family of all fuzzy (i, j) preopen sets in a FBTS X is denoted by (i, j) $FPO(X)$.

2.4 Definition: A fuzzy subset λ of a FBTS (X, τ_i, τ_j) is called fuzzy (i, j) γ-open if $\lambda \wedge \mu$ is fuzzy (i, j) preopen for every fuzzy (i, j) preopen set μ in X. A fuzzy subset η of X is called fuzzy (i, j) γ-closed set if its complement, $1_X - \eta$ in X is a fuzzy (i, j) γ-open set. The collections of all fuzzy (i, j) γ-open sets and fuzzy (i, j) γ-closed sets are denoted by (i, j) $F\gamma\text{-}O(X)$ and (i, j) $F\gamma\text{-}C(X)$ respectively.

2.5 Definition: For a fuzzy set λ in X

(i) (i, j) γ-closure of λ is the intersection of all fuzzy (i, j) γ-closed sets containing λ and is denoted simply as (i, j) $cl_\gamma(\lambda)$.

(ii) (i, j) γ-interior of λ is the union of all fuzzy (i, j) γ-open sets contained in λ and is denoted by (i, j) $int_\gamma(\lambda)$.

2.6 Definition: A fuzzy set λ in a FBTS (X, τ_i, τ_j) is said to be (i, j) generalized fuzzy closed (in short, (i, j) gf closed) if (i, j) $cl(\lambda) \leq \mu$, whenever, μ is a fuzzy (i, j) open set and $\lambda \leq \mu$. A fuzzy set η is called (i, j) generalized fuzzy open

ISSN (Online) 2394-2320

International Journal of Engineering Research in Computer Science and Engineering (IJERCSE)
Vol 5, Issue 2, February 2018

(in short (i,j) gf open) if its complement $1_X - \eta$ is a (i,j) gf closed set. The family of all (i,j) gf closed set is denoted by (i,j) $gfC(X)$.

2.7 Definition: A fuzzy set λ in a FBTS (X,τ_i,τ_j) is said to be a (i,j) γ-generalized fuzzy closed (in short, (i,j) γ-gf closed) set if $\lambda \le \mu \Rightarrow (i,j)\, cl(\lambda) \le \mu$, whenever, λ is a fuzzy $(i,j)\,\gamma$-open set. A fuzzy set η is called (i,j) γ-generalized fuzzy open (in short (i,j) γ-gf open) if its complement $1_X - \eta$ is a (i,j) γ-gf closed set. The family of all (i,j) γ-gf closed set is denoted by (i,j) γ-$gfC(X)$.

2.8 Remark: A fuzzy (i,j) closed set in a FBTS is independent of fuzzy (i,j) γ-closed set, which can be verified from the following example.

2.9 Example: Let us consider a FBTS (X,τ_i,τ_j) with $X = \{x\}$, $\tau_i = \{0_X, 1_X, \{(x,0.2)\}, \{(x,0.7)\}\}$ and $\tau_j = \{0_X, 1_X\}$. Then $(i,j)\, FO(X) = \{0_X, 1_X, \{(x,0.2)\}, \{(x,0.7)\}\}$ and $(i,j)\, FC(X) = \{0_X, 1_X, \{(x,0.3)\}, \{(x,0.7)\}\}$. So $(i,j)\, F\gamma\text{-}O(X) = \{0_X, 1_X, \{(x,\alpha): \alpha \le 0.2$ or $0.3 < \alpha \le 0.7$ or $\alpha > 0.8\}\}$ and $(i,j)\, F\gamma\text{-}C(X) = \{0_X, 1_X, \{(x,\alpha): \alpha < 0.2$ or $0.3 \le \alpha < 0.7$ or $\alpha \ge 0.8\}\}$. Here $\{(x,0.4)\}$ is a fuzzy (i,j) γ-closed set but it is not a fuzzy (i,j) closed set.

2.10 Example: Let us take a FBTS (X,τ_i,τ_j) with $X = \{x,y\}$, $\tau_i = \{0_X, 1_X, \{(x,0.6),(y,0.3)\}, \{(x,0.3),(y,0.3)\}\}$, $\tau_j = \{0_X, 1_X, \{(x,0.3),(y,0.7)\}, \{(x,0.6),(y,0.7)\}\}$. Then $(i,j)\, FO(X) = \{0_X, 1_X, \{(x,0.6),(y,0.3)\}, \{(x,0.3),(y,0.3)\}, \{(x,0.3),(y,0.7)\}, \{(x,0.6),(y,0.7)\}\}$ and $(i,j)\, FO(X) = \{0_X, 1_X, \{(x,0.4),(y,0.7)\}, \{(x,0.7),(y,0.7)\}, \{(x,0.7),(y,0.3)\}, \{(x,0.4),(y,0.3)\}\}$. Thus $(i,j)\, F\gamma\text{-}O(X) = \{0_X, 1_X, \{(x,\alpha),(y,\beta)\}: \alpha > 0.7, \beta > 0.7\}$ and (i,j) $F\gamma\text{-}C(X) = \{0_X, 1_X, \{(x,\alpha),(y,\beta)\}: \alpha < 0.3, \beta < 0.3\}$. Here $\{(x,0.7),(y,0.7)\}$ is a fuzzy (i,j) closed set but it is not a fuzzy (i,j) γ-closed set.

2.11 Theorem: Every fuzzy (i,j) closed (resp. fuzzy (i,j) open) set is (i,j) gf closed (resp. (i,j) gf open).
Proof: Let λ be a fuzzy (i,j) closed set in a given FBTS (X,τ_i,τ_j) and $\lambda \le \mu$, where μ is a fuzzy (i,j) open set in X. Now, $cl(\lambda) = \lambda \le \mu$, since λ is fuzzy (i,j) closed. Therefore λ is (i,j) gf closed.

2.12 Remark: Converse of the above theorem may not be true in general.

2.13 Example: We consider the FBTS taken in example 2.9. Here (i,j) $gfC(X) = \{0_X, 1_X, \{(x,\alpha): 0.2 \le \alpha < 0.3$ or $\alpha > 0.7\}$. Obviously $\{(x,0.25)\}$ is a (i,j) gf closed set but it is not a fuzzy (i,j) closed set.

2.14 Remark: The notions of fuzzy (i,j) γ-closed set and (i,j) gf closed set are independent concepts. This can be observed from the following mentioned example.

2.15 Example: Again we consider the FBTS taken in example 2.9. Now from example 2.13, it is obvious that $\{(x,0.1)\}$ is not a (i,j) gf closed set and $\{(x,0.75)\}$ is a (i,j) gf closed set in X. But from example 2.9, we see that $\{(x,0.1)\}$ is a fuzzy (i,j) γ-closed set and $\{(x,0.75)\}$ is not a fuzzy (i,j) γ-closed set. Thus our claim is verified.

2.16 Theorem: Every fuzzy (i,j) closed set is a (i,j) γ-gf closed set.
Proof: The (i,j) closure of a fuzzy (i,j)closed set is the same set. From the above statement the proof of the theorem is obvious.

2.17 Remark: However the converse of the above theorem is not true, in general, as we see the following example.

2.18 Example: We take the same FBTS like example 2.9. Here $(i,j)\,\gamma\text{-}gfC(X) = \{0_X, 1_X, \{(x,\alpha): 0.2 < \alpha \le 0.3$ or $0.7 < \alpha \le 0.8\}\}$. Clearly the fuzzy set $\{(x,0.75)\}$ is a $(i,j)\,\gamma$-gf closed set but it is not a fuzzy (i,j) closed set.

2.19 Remark: Both (i,j) gf closed set and (i,j) γ-gf closed set are independent of each other. We demonstrate this result by the following examples.

2.20 Example: We take the same FBTS like example 2.9. Now observing example 2.13 and example 2.18, we see that the fuzzy set $\{(x,0.9)\}$ is a (i,j) gf closed set, though it is not a (i,j) γ-gf closed set.

2.21 Example: Let us take a FBTS (X,τ_i,τ_j) with $X = \{x,y\}$, $\tau_i = \{0_X, 1_X, \{(x,0.2),(y,0.3)\}, \{(x,0.4),(y,0.4)\}\}$ and $\tau_j = \{0_X, 1_X, \{(x,0.6),(y,0.7)\}\}$. Then, we have (i,j) $FO(X) = \{0_X, 1_X, \{(x,0.2),(y,0.3)\}, \{(x,0.4),(y,0.4)\}, \{(x,0.6),(y,0.7)\}\}$ and so $(i,j)\, FC(X) = \{0_X 1_X, \{(x,0.8),(y\ 0.7)\}, \{(x,0.6),(y,0.6)\}, \{(x,0.4),(y,0.3)\}\}$. Thus (i,j) $F\gamma\text{-}O(X) = \{0_X, 1_X, \{(x,\alpha),(y,\beta)\}: \alpha > 0.2, \beta > 0.7\}$ and $(i,j)F\gamma\text{-}C(X) = \{0_X, 1_X, \{(x,\alpha),(y,\beta)\}: \alpha < 0.8, \beta < 0.3\}$. Now we suppose $\mu = \{(x,0.2),(y,0.3)\}$. Clearly μ is a (i,j) γ-gf closed set. Again μ is itself a fuzzy (i,j) open set but

ISSN (Online) 2394-2320

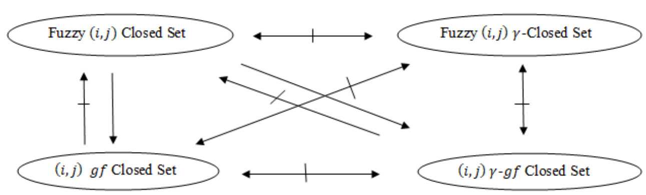
International Journal of Engineering Research in Computer Science and Engineering
(IJERCSE)
Vol 5, Issue 2, February 2018

$(i,j)\ cl(\mu) = \{(x, 0.4), (y, 0.3)\} \nleq \mu$. Therefore μ is not a $(i,j)\ gf$ closed set.

2.22 Remark: The notions of $(i,j)\ \gamma\text{-}gf$ closed set and fuzzy $(i,j)\ \gamma$-closed set are independent of each other.

2.23 Example: Once again, we take the same FBTS like example 2.9. Here $\mu = \{(x, 0.1)\}$ is a fuzzy $(i,j)\ \gamma$-closed set, but from example 2.18, it is obvious that μ is not a $(i,j)\ \gamma\text{-}gf$ closed set. Again from example 2.18, we see that the fuzzy set $\{(x, 0.75)\}$ is a $(i,j)\ \gamma\text{-}gf$ closed set but it is not a fuzzy $(i,j)\ \gamma$-closed set.

2.24 Diagram: From the above remarks and examples we can express the discussed interrelationships as follows:



2.25 Theorem: If λ and η are $(i,j)\ \gamma\text{-}gf$ closed sets, then, $\lambda \vee \eta$ is $(i,j)\ \gamma\text{-}gf$ closed.
Proof: Let $\lambda \vee \eta \leq \mu$ and μ be fuzzy $(i,j)\ \gamma$-open. Then, $\lambda,\ \eta \leq \mu$ and thus, $(i,j)\ cl(\lambda),\ (i,j)\ cl(\eta) \leq \mu$. Hence, $(i,j)\ cl(\lambda \vee \eta) = (i,j)\ cl(\lambda)\ \vee\ (i,j)\ cl(\eta) \leq \mu$ and thus, $\lambda \vee \eta$ is a $(i,j)\ \gamma\text{-}gf$ closed set.

2.26 Remark: The intersection of two $(i,j)\ \gamma\text{-}gf$ closed sets may not be a $(i,j)\ \gamma\text{-}gf$ closed set, which is illustrated in the following example.

2.27 Example: We consider a FBTS (X, τ_i, τ_j) with $X = \{x, y\}, \tau_i = \{0_X, 1_X, \{(x, 0.2), (y, 0)\}\}$ and $\tau_j = \{0_X, 1_X, \{(x, 0), (y, 0.3)\}, \{(x, 0.2), (y, 0.3)\}\}$. Then, (i,j)
$FO(X) = \{0_X, 1_X, \{(x, 0.2), (y, 0)\}, \{ (x, 0), (y, 0.3)\}, \{(x, 0.2), (y, 0.3)\}\}$ and so $(i,j)\ FC(X) = \{0_X, 1_X, \{(x, 0.8), (y, 1)\}, \{(x, 1), (y, 0.7)\}, \{(x, 0.8), (y, 0.7)\}\}$. Thus $(i,j)\ F\gamma\text{-}O(X) = \{0_X, 1_X, \{(x, \alpha), (y, \beta)\}: \alpha \leq 0.2, \beta \leq 0.3$ or $\alpha > 0.8, \beta > 0.7\}$ and $(i,j)\ F\gamma\text{-}\ C(X) = \{0_X, 1_X, \{(x, \alpha), (y, \beta)$: $\alpha \geq 0.8, \beta \geq 0.7$ or $\alpha < 0.2, \beta < 0.3\}$. Now let us take two fuzzy sets $\lambda = \{(x, 0.8), (y, 0.1)\}, \eta = \{(x, 0.1), (y, 0.5)\}$. Obviously both λ and η are $(i,j)\ \gamma\text{-}gf$ closed set. Now $\lambda \wedge \eta = \{(x, 0.1), (y, 0.1)\}$, which is itself a fuzzy $(i,j)\ \gamma$-open set. But $(i,j) cl(\lambda \wedge \eta) = \{(x, 0.8), (y, 0.7)\} \nleq \lambda \wedge \eta$. Therefore $\lambda \wedge \eta$ is not a $(i,j)\ \gamma\text{-}gf$ closed set in X.

2.28 Theorem: If λ is a $(i,j)\ \gamma\text{-}gf$ closed set and $\lambda \leq \eta \leq (i,j)\ cl(\lambda)$, then η is a $(i,j)\ \gamma\text{-}gf$ closed set.
Proof: Let μ be a fuzzy $(i,j)\ \gamma$-open set such that $\eta \leq \mu$. Since, $\lambda \leq \eta \leq \mu$ and λ is a $(i,j)\ \gamma\text{-}gf$ closed set, $(i,j)\ cl(\lambda) \leq \mu$. But $(i,j)\ cl(\eta) \leq (i,j) cl(\lambda)$, since $\eta \leq (i,j) cl(\lambda)$ and thus $(i,j)\ cl(\eta) \leq \mu$. Hence, η is a $(i,j)\ \gamma\text{-}gf$ closed set.

2.29 Theorem: The following statements are equivalent:

(a) λ is a $(i,j)\ \gamma\text{-}gf$ open set in X.
(b) $\eta \leq (i,j)\ int(\lambda)$, whenever η is a fuzzy $(i,j)\ \gamma$-closed set and $\eta \leq \lambda$.

Proof: $(a) \Rightarrow (b)$ Suppose λ be a $(i,j)\ \gamma\text{-}gf$ open set and η be a fuzzy $(i,j)\ \gamma$-closed set such that $\eta \leq \lambda$. Therefore, $1_X - \eta$ is fuzzy $(i,j)\ \gamma$-open and $1_X - \lambda \leq 1_X - \eta$ Now since, $1_X - \lambda$ is a $(i,j)\ \gamma\text{-}gf$ closed set, $1_X -\ (i,j)\ int(\lambda) = (i,j)\ cl(1_X - \lambda) \leq 1_X - \eta$, which implies that $\eta \leq (i,j)\ int(\lambda)$.

$(b) \Rightarrow (a)$ We consider a fuzzy set λ such that $\eta \leq (i,j)\ int(\lambda)$, whenever η is a fuzzy $(i,j)\ \gamma$-closed set and $\eta \leq \lambda$. Here, we have to prove that $1_X - \lambda$ is $(i,j)\ \gamma\text{-}gf$ closed. Let $1_X - \lambda \leq \mu$, where μ is a fuzzy $(i,j)\ \gamma$-open set. Since $1_X - \lambda \leq \mu, 1_X - \mu \leq \lambda$. Hence, by the hypothesis, we have $1_X - \mu \leq (i,j)\ int(\lambda)$ i.e., $1_X - (i,j)\ int(\lambda) \leq \mu$. Thus, $(i,j)\ cl(1_X - \lambda)\ \leq \mu$, which implies that $1_X - \lambda$ is a $(i,j)\ \gamma\text{-}gf$ closed set. Hence λ is a $(i,j)\ \gamma\text{-}gf$ open set in X.

2.30 Theorem: If λ and η are $(i,j)\ \gamma\text{-}gf$ open sets with $\wedge (i,j)\ cl(\eta)\ = \eta\ \wedge (i,j)\ cl(\lambda)\ =\ 0_X$, then $\lambda \vee \eta$ is a $(i,j)\ \gamma\text{-}gf$ open set.
Proof: Suppose μ be a $(i,j)\ \gamma\text{-}gf$ closed set such that $\mu \leq \lambda \vee \eta$. Then, $\mu \wedge (i,j)cl(\lambda) \leq \lambda$, since $\eta \wedge (i,j)cl(\lambda) \leq \lambda$ and hence by theorem 2.25, $\mu \wedge (i,j)cl(\lambda) \leq (i,j)\ int(\lambda)$. In a similar manner, $\mu\ \wedge (i,j)cl(\eta) \leq (i,j)\ int(\eta)$. Thus, we get $\mu = \mu \wedge (\lambda \vee \eta) \leq (\mu \wedge (i,j)cl(\lambda)) \vee \leq (\mu \wedge (i,j)cl(\eta)) \leq (i,j)\ int(\lambda)\ \ \vee (i,j)\ int(\eta) \leq (i,j)\ int(\lambda \vee \eta)$. Therefore, $\mu \leq (i,j)\ int(\lambda \vee \eta)$. Then, by theorem 2.25, $\lambda \vee \eta$ is a $(i,j)\ \gamma\text{-}gf$ open set.

2.31 Theorem: If $(i,j)\ int(\lambda)\ \leq \eta \leq \lambda$ and λ is a $(i,j)\ \gamma\text{-}gf$ open set then, η is also a $(i,j)\ \gamma\text{-}gf$ open set therein.
Proof: Using the theorem 2.28, it can be easily obtained and hence omitted.

ISSN (Online) 2394-2320

International Journal of Engineering Research in Computer Science and Engineering (IJERCSE)
Vol 5, Issue 2, February 2018

III (i,j) γ-GENERALIZED FUZZY CONTINUOUS FUNCTIONS AND RELATED RESULTS

3.1 Definition: A function $f : X \to Y$ from a FBTS (X, τ_i, τ_j) to another FBTS (Y, σ_i, σ_j) is said to be (i,j) fuzzy continuous if the inverse image of every fuzzy (i,j) open set in Y is fuzzy (i,j) open in X.

Following the definition of pairwise γ-continuity due to B. C. Tripathy et al. [6], we introduce (i,j) fuzzy γ-continuous function as follows:

3.2 Definition: A function $f : X \to Y$ from a FBTS (X, τ_i, τ_j) to another FBTS (Y, σ_i, σ_j) is called a (i,j) fuzzy γ-continuous (in short, (i,j) $f\gamma$-continuous) function if the inverse image of every fuzzy (i,j) closed set in Y is fuzzy (i,j) γ-closed in X.

3.3 Definition: A function $f : X \to Y$ from a FBTS (X, τ_i, τ_j) to another FBTS (Y, σ_i, σ_j) is said to be (i,j) generalized fuzzy continuous (in short, (i,j) gf continuous) if the inverse image of every fuzzy (i,j) closed set in Y is (i,j) gf closed in X.

3.4 Definition: Any function f from a FBTS (X, τ_i, τ_j) to another FBTS (Y, σ_i, σ_j) is said to be (i,j) γ-generalized fuzzy continuous (in short, (i,j) γ-gf continuous) function if the inverse image of every fuzzy (i,j) closed set in Y is a (i,j) γ-gf closed set in X.

3.5 Theorem: If λ is a (i,j) γ-gf closed set in a FBTS (X, τ_i, τ_j) and if $f : X \to Y$ is (i,j) $f\gamma$-continuous and fuzzy (i,j) closed, then $f(\lambda)$ is (i,j) γ-gf closed in X.
Proof: Let η be a fuzzy (i,j) open set in Y such that $f(\lambda) \leq \eta$. Then, $\lambda \leq f^{-1}(\eta)$. Since, λ is (i,j) γ-gf closed and $f^{-1}(\eta)$ is fuzzy (i,j) open, (i,j) $cl(\lambda) \leq f^{-1}(\eta)$, i.e., $f\big((i,j)\ cl(\lambda)\big) \leq \eta$. Also, f is fuzzy (i,j) closed and $(i,j)\ cl\big(f(\lambda)\big) \leq (i,j)\ cl(f((i,j)\ cl(\lambda)) = f((i,j)\ cl(\lambda)) \leq \eta$ and hence $f(\lambda)$ is a (i,j) γ-gf closed set.

3.6 Theorem: Every (i,j) fuzzy continuous function defined on a FBTS is (i,j) gf continuous function.

3.7 Remark: The converse is not true, as we see the following example.

3.8 Example: Let us consider two FBTS (X, τ_i, τ_j) and (Y, σ_i, σ_j) with $X = \{x, y\}, Y = \{z, u\}, \tau_i = \{0_X, 1_X, \{(x, 0.2), (y, 0.3)\}\}, \tau_j = \{0_X, 1_X, \{(x, 0.3), (y, 0.4)\}\}, \sigma_i = \{0_Y, 1_Y\}$ and $\sigma_j = \{0_Y, 1_Y, \{(z, 0.8), (u, 0.7)\}, \{(z, 0.1), (z, 0.2)\}\}$. Also we consider a function $f : X \to Y$ such that $f(x) = z, f(y) = u$. Now we have $(i,j)\ FO(X) = \{0_X, 1_X, \{(x, 0.2), (y, 0.3)\}, \{(x, 0.3), (y, 0.4)\}\}$ and $(i,j)\ FC(X) = \{0_X, 1_X, \{(x, 0.8), (y, 0.7)\}, \{(x, 0.7), (y, 0.6)\}\}$. Thus $(i,j)\ gfC(X) = \{0_X, 1_X, \{(x, \alpha), (y, \beta)\}: \alpha \leq 0.3, \beta \leq 0.4$ or $\alpha > 0.8, \beta > 0.7\}$. Evidently f is a (i,j) gf continuous function. But $f^{-1}\{(z, 0.9), (u, 0.8)\} = \{(x, 0.9), (y, 0.8)\} \notin (i,j)\ FC(X)$. Hence f is not a (i,j) fuzzy continuous function.

3.9 Remark: Both the notions of (i,j) gf continuous function and (i,j) γ-gf continuous functions are independent of each other. This result is clearly evident in the following examples.

3.10 Example: Let us suppose that (X, τ_i, τ_j) and (Y, σ_i, σ_j) be two FBTS, where $X = \{x\}, Y = \{z\}, \tau_i = \{0_X, 1_X, \{(x, 0.2)\}\}$, $\tau_j = \{0_X, 1_X, \{(x, 0.8)\}\}$, $\sigma_i = \{0_Y, 1_Y, \{(z, 0.4)\}\}$ and $\sigma_j = \{0_Y, 1_Y, \{(z, 0.3)\}\}$. We define a function $f : X \to Y$ in such a way that, $f(x) = z$. Now, $(i,j)\ FO(X) = \{0_X, 1_X, \{(x, 0.2)\}, \{(x, 0.8)\}\}$ and $(i,j)\ FC(X) = \{0_X, 1_X, \{(x, 0.8)\}, \{x, 0.2)\}\}$. Also $(i,j)\ FO(Y) = \{0_Y, 1_Y, \{(z, 0.4)\}, \{(z, 0.3)\}$ and $(i,j)\ FC(Y) = \{0_Y, 1_Y, \{(z, 0.6)\}, \{(z, 0.7)\}\}$. Here, every fuzzy set in X is fuzzy (i,j) γ-open set as well as (i,j) gf closed set. So, obviously f is a (i,j) gf continuous mapping. Again, $\{(z, 0.7)\}$ is a fuzzy (i,j) closed set in Y. Suppose $f^{-1}\{(z, 0.7)\} = \{(x, 0.7)\} = \lambda$. Now $\lambda \leq \{(x, 0.75)\}$, which is a fuzzy (i,j) γ-open set, but $(i,j)\ cl(\lambda) = \{(x, 0.8)\} \nleq \{(x, 0.75)\}$. So, λ is not a (i,j) γ-gf closed set. So, f is not a (i,j) γ-gf continuous function.

3.11 Example: Let us suppose that (X, τ_i, τ_j) and (Y, σ_i, σ_j) be two FBTS, where $X = \{x, y\}, Y = \{z, u\}, \tau_i = \{0_X, 1_X, \{(x, 0.6), (y, 0.3)\}, \{(x, 0.6), (y, 0.7)\}\}, \tau_j = \{0_X, 1_X, \{(x, 0.3), (y, 0.3)\}, \{(x, 0.3), (y, 0.7)\}\}, \sigma_i = \{0_Y, 1_Y\}$ and $\sigma_j = \{0_Y, 1_Y, \{(z, 0.5), (u, 0.5)\}\}$. Also we consider a function $f : X \to Y$ such that $f(x) = z, f(y) = u$. Now we have $(i,j)\ FO(X) = \{0_X 1_X, \{(x, 0.6), (y, 0.3)\}, \{(x, 0.6), (y, 0.7)\}, \{(x, 0.3), (y, 0.3)\}, \{(x, 0.3), (y, 0.7)\}\}$ and $(i,j) FC(X) = 0_X, 1_X, \{(x, 0.4), (y, 0.7)\}, \{(x, 0.4), (y, 0.3)\}, \{(x, 0.7), (y, .7)\}, \{(x, 0.7), (y, 0.3)\}\}$. Also $(i,j)\ FO(Y) = \{0_Y, 1_Y, \{(z, 0.5), (u, 0.5)\}\}$ and $(i,j)\ FC(Y) = \{0_Y, 1_Y, \{(z, 0.5), (u, 0.5)\}\}$. Now obviously f is a (i,j) γ-gf continuous mapping. Now $\{(z, 0.5), (u, 0.5)\}$ is a fuzzy (i,j) closed set in Y.

ISSN (Online) 2394-2320

International Journal of Engineering Research in Computer Science and Engineering
(IJERCSE)
Vol 5, Issue 2, February 2018

Suppose $\lambda = f^{-1}\{(z, 0.5), (u, 0.5)\} = \{(x, 0.5), (y, 0.5)\}$, which is not a (i,j) gf closed set in X. Therefore f is not a (i,j) gf continuous function.

3.12 Definition: A fuzzy set λ in a FBTS (X, τ_i, τ_j) is called (i,j) γ-generalized fuzzy q-neighbourhood (in short, (i,j) γ-gf q-nbd) of a fuzzy point x_p if there is a (i,j) γ-gf open set μ such that $x_p q \mu \leq \lambda$.

3.13 Definition: For any fuzzy set λ in a FBTS (X, τ_i, τ_j), (i,j) γ -generalized fuzzy closure operator (i,j) cl_γ^* is given as:
(i,j) $cl_\gamma^*(\lambda) = \wedge \{ \eta : \lambda \leq \eta, \eta$ is a (i,j) γ-gf closed set in $X\}$.

3.14 Theorem: Let λ be a fuzzy set in a FBTS (X, τ_i, τ_j) and also let x_p be a fuzzy point in X. Then, $x_p \in (i,j)$ $cl_\gamma^*(\lambda)$ if and only if for each (i,j) γ-gf q-nbd μ of x_p, $\mu q \lambda$.
Proof: It can be easily proved using the definition of (i,j) $cl_\gamma^*(\lambda)$ and by the definition of (i,j) γ-gf q-nbd. Hence, it is omitted.

3.15 Theorem: Let $f : (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ be a function from a FBTS (X, τ_i, τ_j) into another FBTS (Y, σ_i, σ_j). Then the following statements are equivalent:
(i) f is a (i,j) γ-gf continuous function
(ii) the inverse image of each fuzzy (i,j) open set in Y is (i,j) γ-gf open set in X.
Proof: It is straightforward from the definition, hence omitted.

3.16 Theorem: Let $f : (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ be a (i,j) γ-gf continuous function, then $f((i,j)\ cl(\lambda)) \leq (i,j)\ cl(f(\lambda))$, for any fuzzy set λ in X.
Proof: Let λ be a fuzzy set in a FBTS (X, τ_i, τ_j). Then, $\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}((i,j)cl(f(\lambda)))$. Obviously, $(i,j)cl(f(\lambda))$ is a fuzzy (i,j) closed set in Y and since f is (i,j) γ-gf continuous, therefore, $f^{-1}((i,j)cl(f(\lambda)))$ is a (i,j) γ-gf closed set. Thus, $(i,j)\ cl_\gamma^*(\lambda) \leq f^{-1}((i,j)cl(f(\lambda)))$ and hence, $f\left((i,j)\ cl_\gamma^*(\lambda)\right) \leq (i,j)cl(f(\lambda))$.

3.17 Theorem: For any function $f : (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ from a FBTS (X, τ_i, τ_j) into another FBTS (Y, σ_i, σ_j), the following statements are equivalent:

(i) For each fuzzy point x_p in X and each (i,j) fuzzy open q-nbd of $f(x_p)$, there exists a (i,j) γ-gf open q-nbd μ of x_p such that $f(\mu) \leq \eta$.

(ii) $f\left((i,j)\ cl_\gamma^*(\lambda)\right) \leq (i,j)cl(f(\lambda))$, for a fuzzy subset $\lambda \in X$.

(iii) $(i,j)\ cl_\gamma^*(f^{-1}(\eta)) \leq f^{-1}((i,j)\ cl(\eta))$, for a fuzzy subset $\eta \in Y$.

Proof: $(i) \Rightarrow (ii)$ Let $y_p \in f\left(cl_\gamma^*(\lambda)\right)$ and ν be any (i,j) fuzzy open q-nbd of y_p. Then, there exists $x \in X$ such that $f(x)_p = y_p$ and $x_p \in (i,j)\ cl_\gamma^*(\lambda)$ and by (i), there exists a (i,j) γ-gf open q-nbd μ of x_p such that $f(\mu) \leq \eta$. Since, $x_p \in (i,j)\ cl_\gamma^*(\lambda)$, $\mu q \lambda$ and hence, $\nu q f(\lambda)$. Hence, $y_\alpha = f(x_\alpha) \in (i,j)\ cl(f(\lambda))$.

$(ii) \Rightarrow (iii)$ Let $x_p \in X$ and ν be any (i,j) fuzzy open q-nbd of $f(x_p)$. Now we put $\lambda = f^{-1}(1-\nu)$. Then, $x_p \notin \lambda$. Since, $f\left((i,j)\ cl_\gamma^*(\lambda)\right) \leq (i,j)cl(f(\lambda)) \leq (1-\nu) = \lambda$, which implies that $(i,j)\ cl_\gamma^*(\lambda) = \lambda$. Since, $x_p \notin (i,j)\ cl_\gamma^*(\lambda)$, then there exists a (i,j) fuzzy open q-nbd of x_p such that $\mu q \lambda$ and thus, $f(\mu) \leq f(1_X - \lambda) \leq \nu$.

$(iii) \Rightarrow (i)$ This can be easily proved, hence, omitted.

3.18 Definition: A FBTS (X, τ_i, τ_j) is said to be (i,j) fuzzy $T_{1/2}$ space if every (i,j) gf closed set in X is fuzzy (i,j)closed in X.

3.19 Definition: A FBTS (X, τ_i, τ_j) is said to be (i,j) fuzzy γ-$T_{1/2}$ space if every (i,j) γ-gf closed set in X is fuzzy (i,j) closed in X.

3.20 Remark: Every (i,j) fuzzy γ-$T_{1/2}$ space is (i,j) fuzzy $T_{1/2}$ space but the converse is not true in general. We demonstrate this result by the following example.

3.21 Example: Let us consider a FBTS (X, τ_i, τ_j) such that $X = \{x, y\}$, $\tau_i = \{0_X, 1_X\}$ and $\tau_j = \{0_X, 1_X, \{(x,\alpha), (y,\beta)\}$: $\alpha \leq 1, \beta < 0.5\}$. Thus $(i,j)\ FO(X) = \{0_X, 1_X, \{(x,\alpha), (y,\beta)\}: \alpha \leq 1, \beta < 0.5\}$ and $(i,j)\ FC(X) = \{0_X, 1_X, \{(x,\alpha), (y,\beta)\}: \alpha \geq 0, \beta > 0.5\}$. Then $(i,j)\ F\gamma\text{-}C(X) = \{0_X, 1_X, \{(x,\alpha), (y,\beta)\}: \alpha \leq 1, \beta < 0.5\}$. Clearly X is a (i,j) fuzzy $T_{1/2}$ space since every (i,j) gf closed set in X is fuzzy (i,j)closed in X. Now we consider a fuzzy set $\mu = \{(x, 0.9), (y, 0.5)\} \leq 1_X$. Here $(i,j)\ cl(\mu) = 1_X \leq 1_X$. Thus

ISSN (Online) 2394-2320

International Journal of Engineering Research in Computer Science and Engineering (IJERCSE)

Vol 5, Issue 2, February 2018

μ is a (i,j) γ-gf closed set in X but it is not a fuzzy (i,j) closed set. Therefore, X is not a (i,j) fuzzy γ-$T_{1/2}$ space.

3.22 Theorem: Let $f : X \to Y$ be a function from a FBTS (X, τ_i, τ_j) into another FBTS (Y, σ_i, σ_j) such that f is a (i,j) γ-gf continuous function and also suppose $g : Y \to X$ is a (i,j) fuzzy continuous function. Then, the composition $g \circ f$ is a (i,j) γ-gf continuous function.
Proof: Let μ be any fuzzy (i,j) closed set in X. Since, g is a (i,j) fuzzy continuous function, therefore $g^{-1}(\mu)$ is a fuzzy (i,j) closed set in Y. Again, f is (i,j) γ-gf continuous function and so, $f^{-1}(g^{-1}(\mu))$ is a (i,j) γ-gf closed set in X. Thus, $(g \circ f)^{-1} = f^{-1}(g^{-1}(\mu))$ is a (i,j) γ-gf closed set. Hence, $g \circ f$ is a (i,j) γ-gf continuous function.

3.23 Remark: The composition of any two (i,j) γ-gf continuous functions may not be (i,j) γ-gf continuous. It is verified in the following example.

3.24 Example: We suppose three FBTS (X, τ_i, τ_j), (Y, σ_i, σ_j) and (Z, ρ_i, ρ_j) where $X = \{x\}, Y = \{y\}, Z = \{z\}$, $\tau_i = \{0_X, 1_X, \{(x, 0.3)\}\}, \tau_j = \{0_X, 1_X, \{(x, 0.6)\}, \{(x, 0.8)\},$ $\sigma_i =, \{0_Y 1_Y, \{(y, 0.1)\}\}, \sigma_j = \{0_Y, 1_Y, \{(y, 0.2)\}, \{(y, 0.3)\}\}$ $, \rho_i = \{0_Z, 1_Z, \{(z, 0.4)\}$ and $\rho_j = \{0_Z, 1_Z, \{(z, 0.6)\}\}$. Also we define two functions $f: X \to Y$ and $g: Y \to Z$ such that $f(x) = y$ and $g(y) = z$. Here f and g are both (i,j) γ-gf continuous function, but $g \circ f$ is not a (i,j) γ-gf continuous function.

IV. CONCLUSION

In this paper we studied $(i,j)\,\gamma$-gf closed set in a fuzzy bitopological space in details. We discovered some interesting results unlikely to the cases of other generalizations of fuzzy closed sets available in literature. The relations of rgf-closed set [10], grf-closed set [3], θ-gf closed set [11] with fuzzy closed set and generalized fuzzy closed set are linear in nature, but in our study we found that though every fuzzy (i,j) closed set is a (i,j) gf closed set but (i,j) gf closed set is completely independent of (i,j) γ-gf closed set. Also we see that fuzzy (i,j) closed set and fuzzy (i,j) γ-closed set are independent concepts. Furthermore we introduced various types of continuous functions between two FBTS and examined interrelationships among them. Lastly we established some properties of (i,j) gf continuous functions with the help of (i,j) γ -generalized fuzzy closure operator.

REFERENCES:

[1] A. Kandil, Biproximities and fuzzy bitopological spaces. Simon Stevins, 63, 45-66 (1989).

[2] A. Paul, B. Bhattacharya, J. Chakraborty, On Λ^{γ}-sets in Fuzzy Bitopological Space. Bol. Soc. Paran. Mat., 35(3), 285{299 (2017).

[3] B. Bhattacharya, J. Chakaraborty, Generalized Regular Fuzzy Closed Sets and Their Applications. 23(1), 227-239 (2015).

[4] B. Bhattacharya, A. Paul. On Bitopological γ-open sets, IOSR J. of Mathematics, 5(2), 10-14 (2013).

[5] B. Bhattacharya, Fuzzy Independent Topological Spaces Generated by Fuzzy γ^*-Open Sets and Their Applications. Afrika Mathematika,28(5-6), 909-928 (2017).

[6] B. C. Tripathy, S. Debnath. γ-Open Sets and γ-Continuous Mappings in Fuzzy Bitopological Spaces, J. of Inteliigence and Fuzzy Systems, 24, 631-635 (2013).

[7] G. Balasubramanian, P. Sundaram, On Some Generalizations of Fuzzy Continuous Functions, Fuzzy Sets and Systems. 86, 93-100 (1997).

[8] J. Cao, M. Ganster, I. Reilly, On Generalized Closed Sets, Topology and its Applications, 123, 37-46 (2002).

[9] J. C. Kelly. Bitopological Spaces, Proc. London Math. Soc., 13, 71{89 (1963).

[10] J. H. Park, J. K. Park. On Regular Generalized Fuzzy Closed Sets and Generalization of Fuzzy Continuous Functions, Indian J. Pure Appl. Math., 34(7), 1013-1024 (2003).

[11] M. E. El-Shafei, A. H. Zakari, θ-Generalized Closed Sets in Fuzzy Topological Spaces, Arab Journal for Science and Engineering, 31(2A), 197-206 (2006).

[12] M. L. Thivagar, O. Ravi, On strong forms of $(1,2)^*$ quotient mappings in Bitopological Space, International J. Math. Game Theory and Algebra, 14(6), 481-492, (2004).

[13] N. Levine. Generalized Closed Sets in Topology. Rend. Circ. Math. Palermo., 19, 89-96 (1970).

[14] N. Palaniappan, K. Chandrasekhar Rao, Regular Generalized Closed Sets, Kyungpook Mathematical Journal. 33(2), 211-219 (1993).

[15] P. P. Ming, L. Y. Ming, Fuzzy topology. I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., 76, 571-599 (1980).

[16] S. Bhattacharya, On Generalized Regular Closed Sets, Int. J. Contemp. Math. Sciences, 6(3), 145-152 (2011).

[17] S. S. Kumar, On Fuzzy Pairwise γ-Continuity and Fuzzy Pairwise Pre-continuity, Fuzzy Sets and Systems. 62,231-238 (1994).

[18] T. Fukutake, On Generalized Closed Sets in Bitopological Spaces, Bull. Fukuoka. Univ. Ed. 35(III), 1928 (1986).