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Solution of the Verhulst Model in Mathematical Biology Using Natural Decomposition Method (NDM)

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Abstract - **In this research paper, we apply a novel method called the Natural Decomposition Method (NDM) to the non-linear ordinary differential equation called the Verhulst model or the Logistic growth model. Natural Decomposition Method (NDM) is based on the Natural Transform Method (NTM) and Adomain Decomposition Method (ADM). We try to give an approximate solution to the Verhulst model using natural decomposition method and we also observe the behaviour of the solution obtained. NDM makes it very easy to solve linear and non-linear ordinary differential equations and gives exact solutions in the form of rapid convergence series.**

Keywords: Adomian decomposition method, Laplace transforms, Natural transforms, Natural Decomposition Method, Ordinary differential equations, Verhulst Model.

1. INTRODUCTION

From past few years, researchers are interested in getting analytical and numerical solutions to the non-linear ordinary differential equations. Non-linear differential equations gained huge amount of interest because of its broad applications in many branches of applied mathematics, pure mathematics, problems in engineering, computational sciences, physical and biological sciences etc. Therefore it is very important to be familiar with all the mathematical techniques or methods for solving linear and non-linear ordinary differential equations. Verhulst model is a continuous population model for single species in Mathematical Biology, it is a non-linear model, There are many integral transform methods [3]- [11] exists in the literature to solve ODE's. The Adomian decomposition method(ADM) [19], [20] proposed by George Adomian is also used solve linear and non-linear PDE's. Mahmoud S. Rawashdeh and Shehu Maitama used the N-Transform to solve non-linear ordinary differential equations and showed that the Natural Decomposition Method (NDM).

[1], [2] shows reliable results in supplying exact solutions and analytical approximate solutions that converges to the exact solutions.

In this paper, we use the integral transform method called Natural decomposition method (NDM) [1], [2] to find the approximate solution to the Verhulst model(Logistic growth model) [12]. The rest of the paper is organized as follows: In section II and III we review the concepts of Verhulst model and natural transform method. Section IV we gives overview of NDM. In section V we apply the NDM to the Verhulst model and obtain the solution in the form of series, section VI gives the conclusion of this paper.

II. VERHULST MODEL IN MATHEMATICAL BIOLOGY

Verhulst model [12] is a continuous population model for single species, some examples of single species population are human population, rabbit population, population of an endangered species, bacterial growth and so on. Whatever may be the population the Verhulst model is very useful to understand the dynamic process involved and also in making practical predictions. N(t) be the population of the species at time t, the model looks like

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) \quad \text{(II.1)}$$

This is called logistic growth in a population, here r and k are positive constants and k is called carrying capacity of the environment which is determined by the available sustainable resources. We consider at time t = 0 the population to be

$\boldsymbol{N(0) = N_0}$ the solution to the above equation II.1 is given by

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$$N(t) = \frac{N_0 K e^{(rt)}}{K + N_0(e^{(rt)} - 1)} \to K \ \ as \ \ t \to \infty \qquad \text{(II.2)}$$

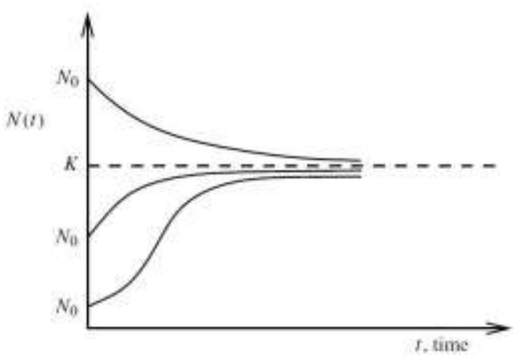

Fig. 1. logistic population growth of Verhulst model

III. NATURAL TRANSFORMATION

Let us consider a function f(t), t ∈ (−∞,∞), then the general integral transform is defined as follows [13], [14]:

$$I[f(t)](s) = \int_{-\infty}^{\infty} K(s,t) f(t)\, dt, \qquad \text{(III.1)}$$

where $k(s, t)$ represents the kernel of the transform, 's' is the real (complex) number which is independent of t. when $k(s, t) = e^{-st}$ then equation III.1 gives the Laplace Transformation.
now for f(t), t ∈ (−∞, ∞) consider the integral transformation defined by:

$$I[f(t)](s,u) = \int_{-\infty}^{\infty} K(s,t) f(ut)\, dt \qquad \text{(III.2)}$$

Definition: The natural transform of the function $f(t)$, $t \in (-\infty, \infty)$ is defined by [13], [14]:

$$\mathbb{N}[f(t)] = R(s,u) = \int_{-\infty}^{\infty} e^{-st} f(ut)\, dt; \quad s, u \in (-\infty, \infty) \qquad \text{(III.3)}$$

We obtained this by putting k(s, t) = e−st in equation III.2. where N[f(t)] is the natural transformation of the time function f(t) and the variables s and u are the natural transform variables. Note that equation III.3 can be written in the form [15], [16]

$$N[f(t)H(t)] = N^{+}[f(t)] = \int e^{-st} f(ut)\, dt\, ;\ s, u \in (0,\infty)$$

$$N[f(t)H(t)] = R^{+}(s, u) = \int e^{-st} f(ut)\, dt\, ;\ s, u \in (0,\infty)$$

where H(t) is the Heaviside function. Note if u = 1, then equation III can be reduced to the Laplace transform.

We now give some important formulas, that is Natural transformations of some functions:

1. when ***f(t)=1*** we have

$$\mathbb{N}^{+}[f(t)] = \int_0^\infty e^{-st} f(ut)\, dt; \qquad s, u \in (0,\infty)$$
$$\mathbb{N}^{+}[1] = \int_0^\infty e^{-st} (1)\, dt; \qquad s, u \in (0,\infty)$$
$$\mathbb{N}^{+}[1] = \tfrac{-1}{s} \left[e^{st}\right]_0^\infty$$
$$\mathbb{N}^{+}[1] = \tfrac{1}{s}$$

2. when ***f(t) = t*** we have

$$\mathbb{N}^{+}[f(t)] = \int_0^\infty e^{-st} f(ut)\, dt; \qquad s, u \in (0,\infty)$$
$$\mathbb{N}^{+}[t] = \int_0^\infty e^{-st} (ut)\, dt; \qquad s, u \in (0,\infty)$$
$$\mathbb{N}^{+}[t] = u\int_0^\infty e^{-st} (t)\, dt; \qquad s, u \in (0,\infty)$$
$$putting \quad st = x \quad wehave \quad dt = \tfrac{dx}{s}$$
$$\mathbb{N}^{+}[t] = \tfrac{u}{s^2} \int_0^\infty e^{-x} (x)^{2-1}\, dx; \qquad s, u \in (0,\infty)$$
$$since \quad \Gamma n = \int_0^\infty e^{-x} (x)^{n-1}\, dx; \quad wehave$$
$$\mathbb{N}^{+}[t] = \tfrac{u}{s^2}\Gamma 2$$
$$\mathbb{N}^{+}[t] = \tfrac{u}{s^2} \qquad since \quad \Gamma n = (n-1)!$$

3. when ***f(t) = e^at*** we have

$$\mathbb{N}^{+}[f(t)] = \int_0^\infty e^{-st} f(ut)\, dt; \qquad s, u \in (0,\infty)$$
$$N^{+}[e^{at}] = \int e^{-st} e^{uat} dt; \qquad s,u \in (0,\infty)$$
$$[e \quad] = \ _0 \ e \qquad ; \qquad s, u \quad (0,\infty)$$
$$\mathbb{N}^{+}[e^{at}] = \tfrac{-1}{s-au} \left[e^{-t(s-au)}\right]_0^\infty$$
$$\mathbb{N}^{+}[e^{at}] = \tfrac{1}{s-au}$$

4. when ***f(t) = t^n*** we have

$$\mathbb{N}^{+}[t^n] = \tfrac{u^n}{s^{n+1}} \int_0^\infty e^{-x} (x)^{n+1-1}\, dx$$
$$\Gamma n = \int_0^\infty e^{-x} (x)^{n-1}\, dx;$$
$$\mathbb{N}^{+}[t^n] = \tfrac{u^n}{s^{n+1}} \Gamma(n+1)$$
$$\mathbb{N}^{+}[t^n] = \tfrac{u^n}{s^{n+1}} (n)!$$
$$\mathbb{N}^{-}\left[\tfrac{t^n}{(n)!}\right] = \tfrac{u^n}{s^{n+1}}$$

5. when ***f(t) = sin(t)***

$$\mathbb{N}^{+}[f(t)] = \int_0^\infty e^{-st} f(ut)\, dt; \qquad s, u \in (0,\infty)$$
$$\mathbb{N}^{+}[sin(t)] = \int_0^\infty e^{-st} sin(ut)\, dt; \qquad s, u \in (0,\infty)$$
$$\mathbb{N}^{+}[sin(t)] = \int_0^\infty e^{-st} \left(\tfrac{e^{iut} - e^{-iut}}{2i}\right) dt; \qquad s, u \in (0,\infty)$$

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$\mathbb{N}^{-}[sin(t)] = \frac{u}{s^2+u^2}$ $s, u \in (0, \infty)$

6. when ***f(t)=cos(t)*** similar we will have
$\mathbb{N}^{-}[cos(t)] = \frac{s}{s^2+u^2}$ s, u ∈ (0, ∞)

the above formulas can shown in a table as shown below [11], [13], [14]:

Table1: Transformations of the function f(t) in Laplace and Natural transformation

$f(t)$	L[$f(t)$]	N[$f(t)$]
1	$\frac{1}{s}$	$\frac{1}{s}$
t	$\frac{1}{s^2}$	$\frac{u}{s^2}$
e^{at}	$\frac{1}{s-a}$	$\frac{1}{s-au}$
$\frac{t^{n-1}}{(n-1)!}, n = 1, 2, 3, ..$	$\frac{1}{s^n}$	$\frac{u^{n-1}}{s^n}$
$sin(t)$	$\frac{1}{1+s^2}$	$\frac{u}{s^2+u^2}$
$cos(t)$	$\frac{s}{1+s^2}$	$\frac{s}{s^2+u^2}$

some properties of Natural Transformation are given below [11], [12], [17], [18]: we have

$$\mathbb{N}^{+}[f(t)] = \int_0^\infty e^{-st} f(ut)\, dt; \quad s, u \in (0, \infty) = R(s, u)$$

or we can write

$$\mathbb{N}^{+}[y(t)] = \int_0^\infty e^{-st} f(ut)\, dt; \quad s, u \in (0, \infty) = Y(s, u)$$

Table2: Properties of Natural transformation

Function	Natural Transform of function
$y(t)$	$Y(s, u)$
$y(at)$	$\frac{1}{a}Y(s, u)$
$\frac{dy(t)}{dt}$	$\frac{s}{u}Y(s, u) - \frac{y(0)}{u}$
$\frac{d^2y(t)}{dt^2}$	$\frac{s^2}{u^2}Y(s, u) - \frac{s}{u^2}y(0) - \frac{y'(0)}{u}$
$\gamma y(t) \pm \beta v(t)$	$\gamma Y(s, u) \pm \beta V(s, u)$

IV. NATURAL DECOMPOSITION METHODOLOGY

In this section, we give a brief idea of the NDM [1], [2]. Given an non-linear ODE with a initial condition, we first apply the N-transformation to the ODE and then substitute the initial condition, now take the inverse N-transformation to get the unknown function. We now assume an infinite series solution to the unknown function and we also represent the non-linear term by an Adomain polynomial. We compute the terms of the infinite series separately by generating a recursive relation. Hence resulting in a solution to the unknown function.
Remark: For more details on NDM please refer to [1], [2].

V. RESULT AND DISCUSSION OF VERHULST MODEL USING NDM:

Now we try to give approximate solution to the Verhulst model using Natural decomposition mehod (NDM) [2]. Let p(t) be the population of the species at time t, then from II.1 we have the nonlinear ordinary differential equation subject to the initial condition p(0) = p0 as follows

$$\frac{dp(t)}{dt} = rp(t)\left(1 - \frac{p(t)}{K}\right) \quad p(0) = p_0 \qquad \text{(V.1)}$$

taking natural transform on both sides of equation V.1 , we get

$$\mathbb{N}^{+}[p'(t)] = \mathbb{N}^{+}[rp] - \mathbb{N}^{+}\left[\frac{r}{k}p^2\right]$$

using the properties of Natural transform from Table2 we have

$$\frac{s}{u}P(s, u) - \frac{p(0)}{u} = r\mathbb{N}^{+}[p(t)] - \frac{r}{k}\mathbb{N}^{+}[p^2(t)] \qquad \text{(V.2)}$$

putting the initial condition in V.2 we have

$$\frac{s}{u}P(s, u) - \frac{p_0}{u} = r\mathbb{N}^{+}[p(t)] - \frac{r}{k}\mathbb{N}^{+}[p^2(t)]$$

$$\frac{s}{u}P(s, u) = \frac{p_0}{u} + r\mathbb{N}^{+}[p(t)] - \frac{r}{k}\mathbb{N}^{+}[p^2(t)]$$

$$\frac{s}{u}P(s, u) - r\mathbb{N}^{+}[p(t)] = \frac{p_0}{u} - \frac{r}{k}\mathbb{N}^{+}[p^2(t)]$$

$$\frac{s}{u}P(s, u) - rP(s, u) - = \frac{p_0}{u} - \frac{r}{k}\mathbb{N}^{+}[p^2(t)]$$

$$P(s, u)\left[\frac{s}{u} - r\right] = \frac{p_0}{u} - \frac{r}{k}\mathbb{N}^{+}[p^2(t)]$$

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$$P(s,u)=\frac{p_0}{u\left[\frac{s}{u}-r\right]}-\frac{r}{k\left[\frac{s}{u}-r\right]}\mathbb{N}^{+}[p^2(t)]$$

$$P(s,u)=\frac{p_0}{(s-ru)}-\frac{1}{\frac{k}{r}\left[\frac{s}{u}-r\right]}\mathbb{N}^{+}[p^2(t)]$$

$$P(s,u)=\frac{p_0}{(s-ru)}-\frac{1}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[p^2(t)] \qquad \text{(V.3)}$$

applying inverse natural transformation to the equation V.3

$$\mathbb{N}^{-1}[P(s,u)]=\mathbb{N}^{-1}\left[\frac{p_0}{(s-ru)}\right]-\mathbb{N}^{-1}\left[\frac{1}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[p^2(t)]\right]$$

$$p(t)=p_0e^{rt}-\mathbb{N}^{-1}\left[\frac{1}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[p^2(t)]\right]$$

$$p(t)=p_0e^{rt}-\frac{1}{k}\mathbb{N}^{-1}\left[\frac{1}{\left(\frac{s}{ur}-1\right)}\mathbb{N}^{+}[p^2(t)]\right] \qquad \text{(V.4)}$$

now we assume the infinite series solution of the unknown function p(t) of the form

$$p(t)=\sum_{n=0}^{\infty}p_n(t) \qquad \text{(V.5)}$$

putting the equation V.5 in the equation V.4 we get

$$\sum_{n=0}^{\infty}p_n(t)=p_0e^{rt}-\frac{1}{k}\mathbb{N}^{-1}\left[\frac{1}{\left(\frac{s}{ur}-1\right)}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty}A_n(t)\right]\right] \qquad \text{(V.6)}$$

$A_n(t)$ is the Adomain polynomial representing the nonlinear term $p^2(t)$. we can generate the recursive relation as follows

$$p_0(t)=p_0e^{rt}$$

$$p_1(t)=\mathbb{N}^{-1}\left[\frac{1}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[A_0(t)]\right]$$

$$p_2(t)=\mathbb{N}^{-1}\left[\frac{1}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[A_1(t)]\right] \qquad \text{(V.7)}$$

thus the general recursive relation is given by

$$p_{n+1}(t)=\mathbb{N}^{-1}\left[\frac{1}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[A_n(t)]\right] \qquad \text{(V.8)}$$

we can compute the remaining components of the unknown function p(t) as follows

$$p_1(t) = \mathbb{N}^{-1}\left[\frac{1}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[A_0(t)]\right]$$

$$p_1(t) = \mathbb{N}^{-1}\left[\frac{1}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[p_0^2(t)]\right]$$

$$p_1(t) = \mathbb{N}^{-1}\left[\frac{1}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[p_0^2e^{2rt}]\right]$$

$$p_1(t) = \mathbb{N}^{-1}\left[\frac{p_0^2}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}[e^{2rt}]\right]$$

$$p_1(t) = \mathbb{N}^{-1}\left[\frac{p_0^2}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}\left[1+\frac{2rt}{1!}+\frac{4r^2t^2}{2!}+..\right]\right]$$

now let us only take the first two terms of the exponential series for our convenience

$$p_1(t)=\mathbb{N}^{-1}\left[\frac{p_0^2}{\left(\frac{sk}{ur}-k\right)}\mathbb{N}^{+}\left[1+\frac{2rt}{1!}\right]\right] \qquad \text{(V.9)}$$

from table2 we can write

$$p_1(t) = \mathbb{N}^{-1}\left[\frac{p_0^2}{\left(\frac{sk}{ur}-k\right)}\left[\frac{1}{s}+\frac{2ru}{s^2}\right]\right]$$

$$p_1(t) = \mathbb{N}^{-1}\left[\frac{p_0^2}{\frac{s^2k}{ur}-ks}+\frac{2rup_0^2}{\frac{s^3k}{ur}-ks^2}\right]$$

now putting $a=\frac{k}{r}$ we the above equation will become

$$p_1(t) = p_0^2\mathbb{N}^{-1}\left[\frac{1}{\frac{as^2}{u}-ks}\right]+2rp_0^2\mathbb{N}^{-1}\left[\frac{u}{\frac{s^3a}{u}-ks^2}\right]$$

$$p_1(t) = p_0^2\mathbb{N}^{-1}\left[\frac{u}{s(as-ku)}\right]+2rp_0^2\mathbb{N}^{-1}\left[\frac{u^2}{s^2(as-ku)}\right]$$

putting the value of $a=\frac{k}{r}$ and taking $\frac{k}{r}$ common in the denominator the above equation will become

$$p_1(t)=\frac{rp_0^2}{k}\mathbb{N}^{-1}\left[\frac{u}{s(s-ur)}\right]+\frac{2r^2p_0^2}{k}\mathbb{N}^{-1}\left[\frac{u^2}{s^2(s-ur)}\right] \qquad \text{(V.10)}$$

we make use of the fact that putting u=1 Natural Transformation becomes Laplace transformation, so the above equationV.10 becomes

$$p_1(t)=\frac{rp_0^2}{k}\mathbb{L}^{-1}\left[\frac{1}{s(s-r)}\right]+\frac{2r^2p_0^2}{k}\mathbb{L}^{-1}\left[\frac{1}{s^2(s-r)}\right] \qquad \text{(V.11)}$$

$$p_1(t)=\frac{rp_0^2}{k}\mathbb{L}^{-1}\left[\frac{\frac{1}{(s-r)}}{s}\right]+\frac{2r^2p_0^2}{k}\mathbb{L}^{-1}\left[\frac{\frac{1}{s(s-r)}}{s}\right] \qquad \text{(V.12)}$$

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putting these in the equation V.12 we get:

$$p_1(t) = \frac{rp_0^2}{k}\int_0^t e^{rt}\,dt + \frac{2r^2p_0^2}{k}\int_0^t \frac{e^{rt}-1}{r}\,dt$$

$$p_1(t) = \frac{p_0^2}{k}\left[e^{rt}\right]_0^t + \frac{2rp_0^2}{k}\left[\frac{e^{rt}}{r} - t\right]_0^t$$

$$p_1(t) = \frac{p_0^2}{k}\left[3e^{rt} - 2rt - 3\right] \qquad \text{(V.13)}$$

(V.14)

now putting V.7 and V.14 in V.5 we get unknown p(t) as shown below

$$p(t) = \sum_{n=0}^{\infty} p_n(t)$$

$$p(t) = p_0e^{rt} + \frac{p_0^2}{k}\left[3e^{rt} - 2rt - 3\right] + \ldots\ldots \qquad \text{(V.15)}$$

Hence the approximate solution the Verhulst model is given by equation V.15 that is

$$p(t) = p_0e^{rt} + \frac{p_0^2}{k}\left[3e^{rt} - 2rt - 3\right] + \ldots\ldots$$

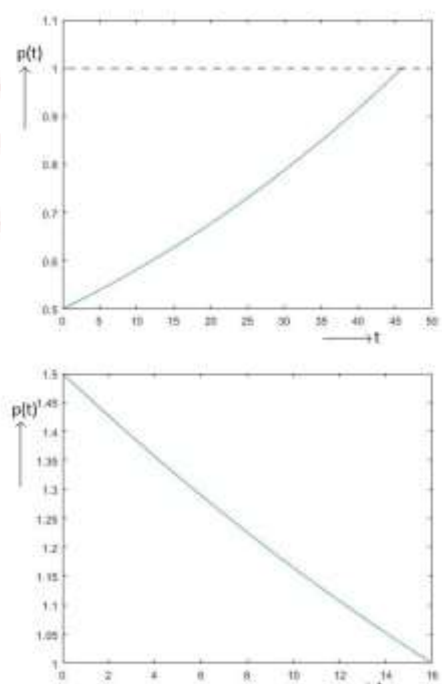

Fig. 2. population growth of Verhulst model: when (a) intial population is less than the carrying capacity (b) intial population is greater than the carrying capacity

The plot to the above equation V.15 represents the population of a single species with time, by taking the carrying capacity k = 1, initial population p0 = 0.5 and the rate of growth r = 0.001.We observe that the population of the species starts increasing with time and approaches the carrying capacity k = 1 when t = 46 (in this case). Here figure 2(b) represents the population decline when the initial population is p0 = 1.5 carrying capacity k = 1 and the rate of growth r = 0.001 it is observe that the population of the species starts decreasing (when the initial population is more than carrying capacity) with time and approaches the carrying capacity k = 1 According to the logistic growth model, as the population reaches the carrying capacity, the growth rate reduces that is the slope of the graph reduces. But, here we observe that there is no decrease in growth rate or in the slope of the graph as the population reaches the carrying capacity, this is because of the approximations we have considered, we ended up with an approximate solution to the Verhulst model.

VI. CONCLUSION

In this paper, we used the Natural decomposition method for solving Verhulst model in Mathematical Biology. We successfully found an approximate solution to the Verhulst model. We feel that the NDM is better over the existing techniques. Our goal in the future is to improve the solution, that is, to get closer to the exact solution and to apply the NDM to other non-linear differential equations that appear in various areas of Mathematical Biology.

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